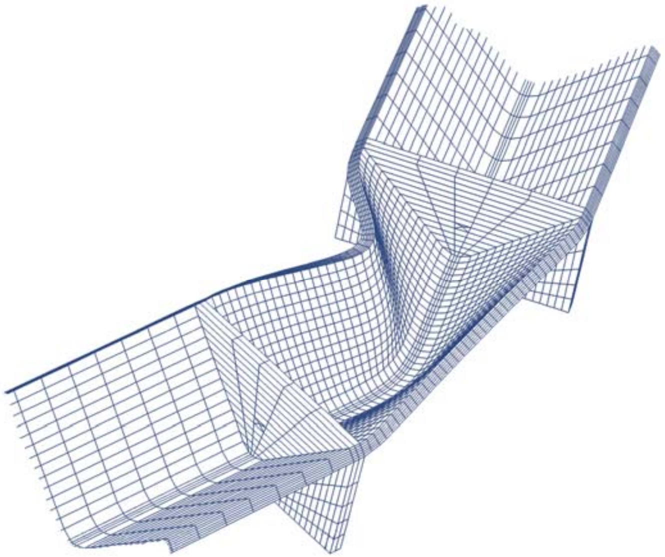
The Finite Element Method for the Analysis of Non-Linear and Dynamic Systems



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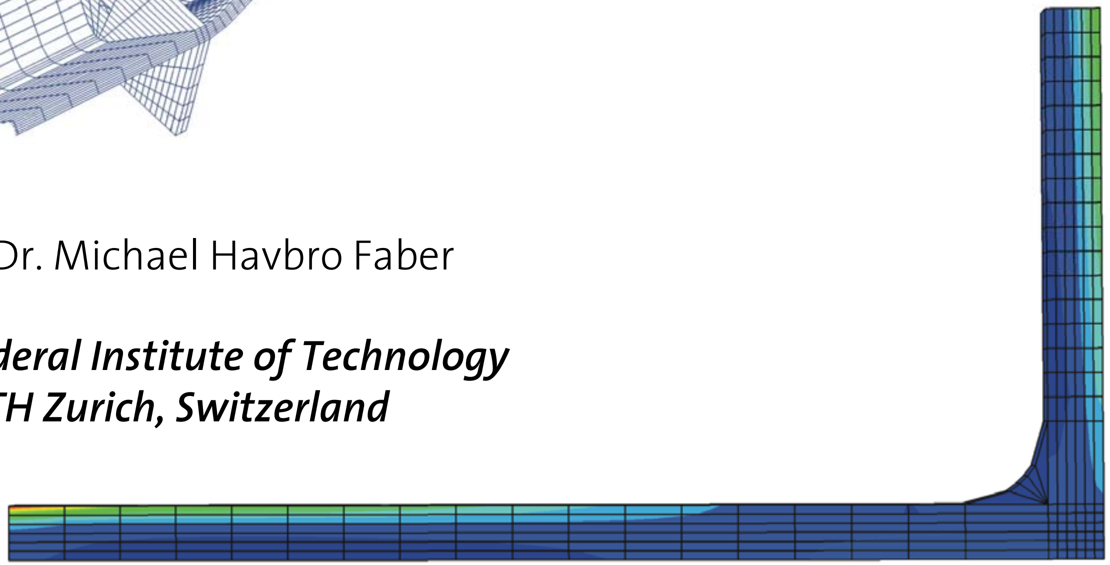
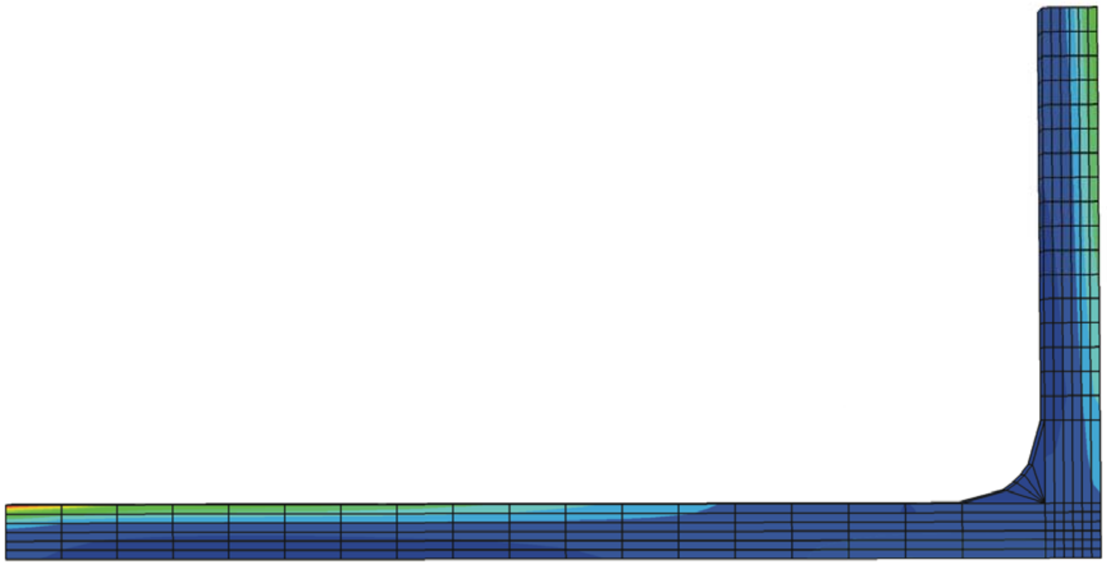
Swiss Federal Institute of Technology
ETH Zurich, Switzerland



Contents of Today's Lecture

- **Motivation, overview and organization of the course**
- **Introduction to non-linear analysis**
- **Formulation of the continuum mechanics incremental equations of motion**



Motivation, overview and organization of the course

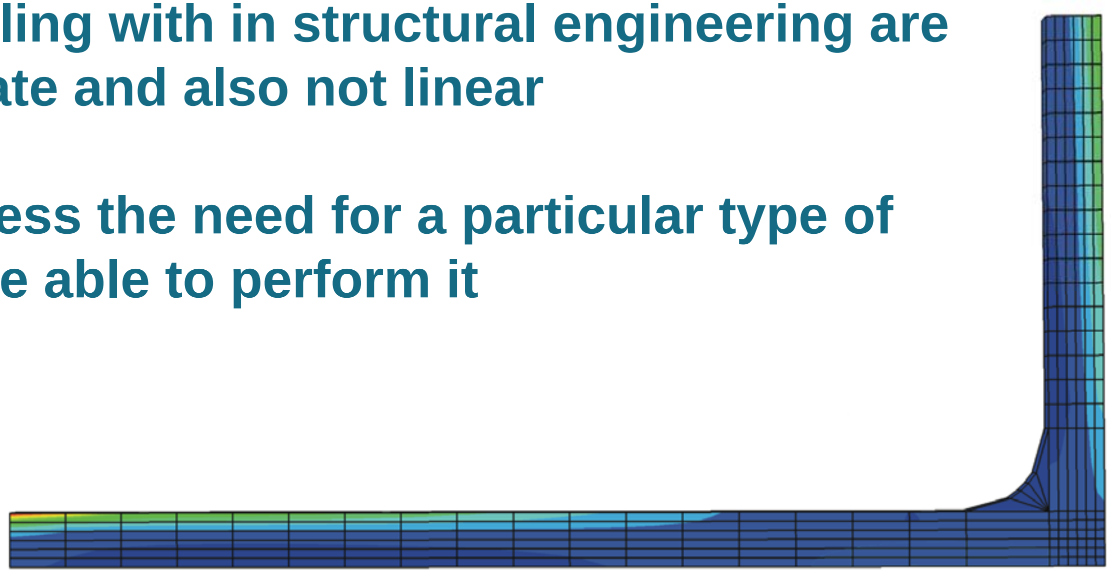
- **Motivation**

In FEM 1 we learned about the steady state analysis of linear systems

however,

the systems we are dealing with in structural engineering are generally not steady state and also not linear

We must be able to assess the need for a particular type of analysis and we must be able to perform it



Motivation, overview and organization of the course

- **Motivation**

What kind of problems are not steady state and linear?

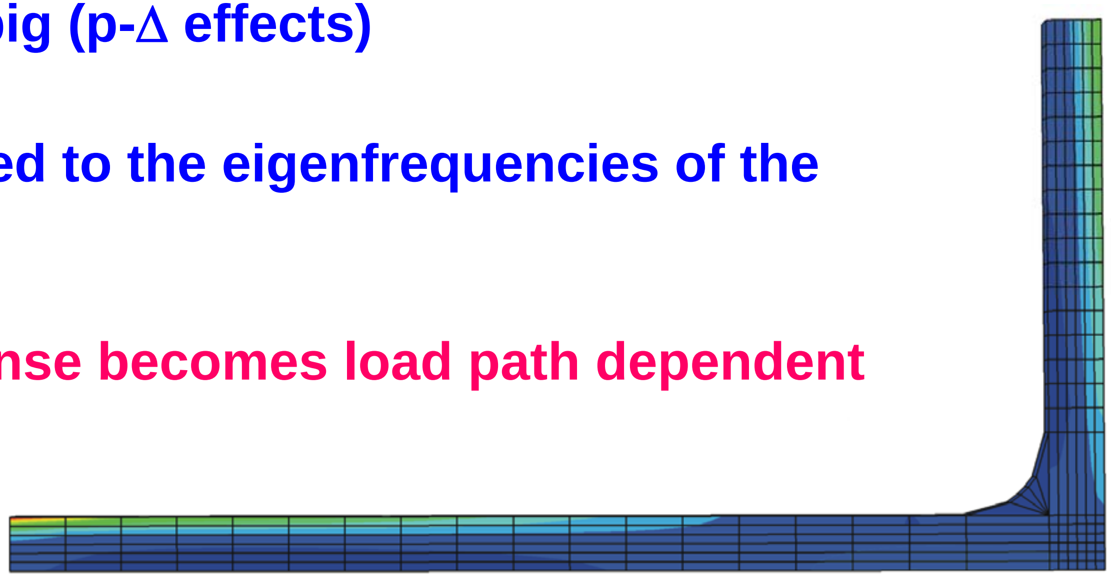
E.g. when the:

material behaves non-linearly

deformations become big (p-Δ effects)

loads vary fast compared to the eigenfrequencies of the structure

General feature: Response becomes load path dependent



Motivation, overview and organization of the course

- **Motivation**

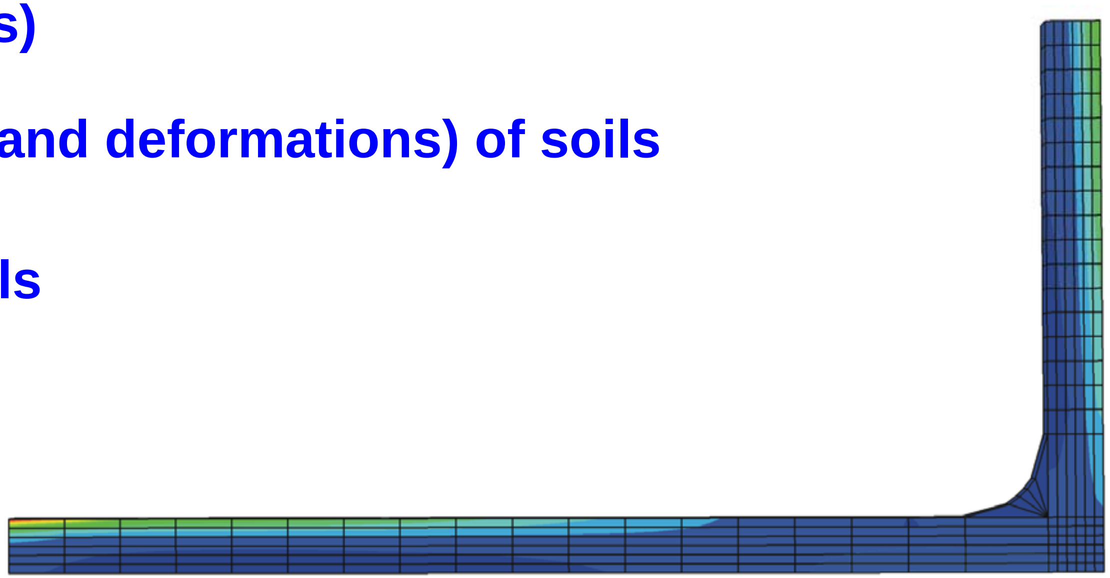
What is the “added value” of being able to assess the non-linear non-steady state response of structures ?

E.g. assessing the:

- structural response of structures to extreme events (rock-fall, earthquake, hurricanes)

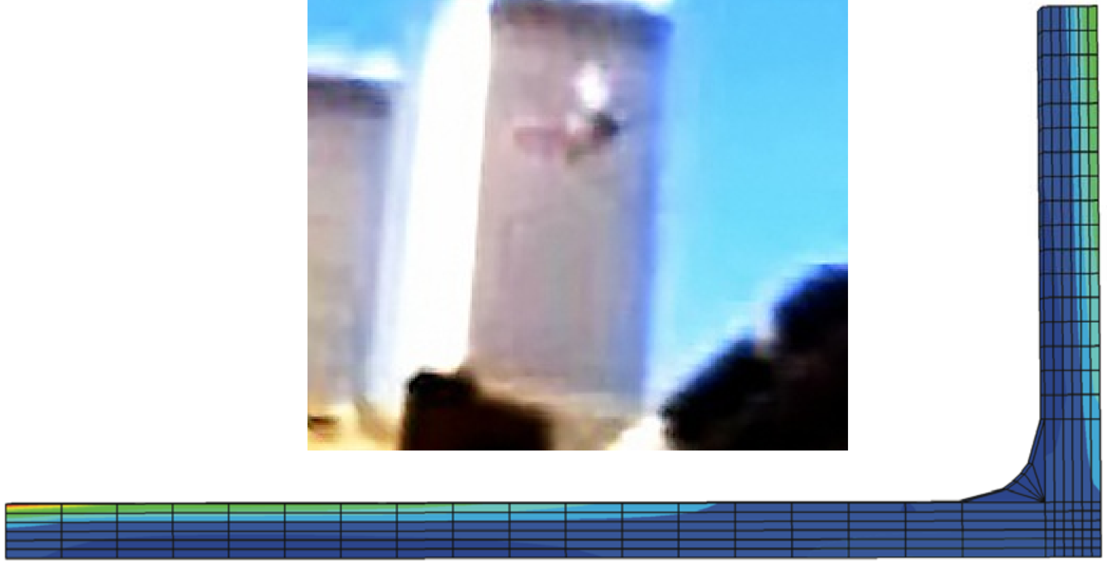
- performance (failures and deformations) of soils

- verifying simple models



Motivation, overview and organization of the course

- **Collapse Analysis of the World Trade Center**



Motivation, overview and organization of the course

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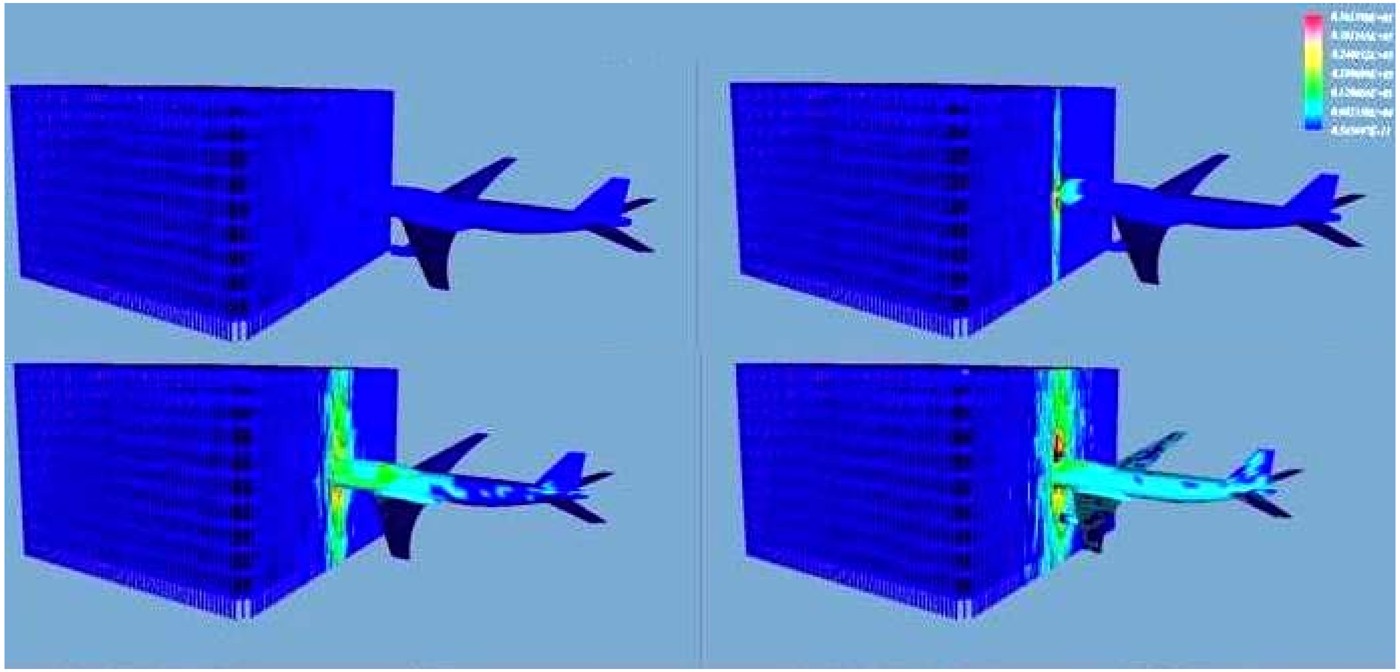
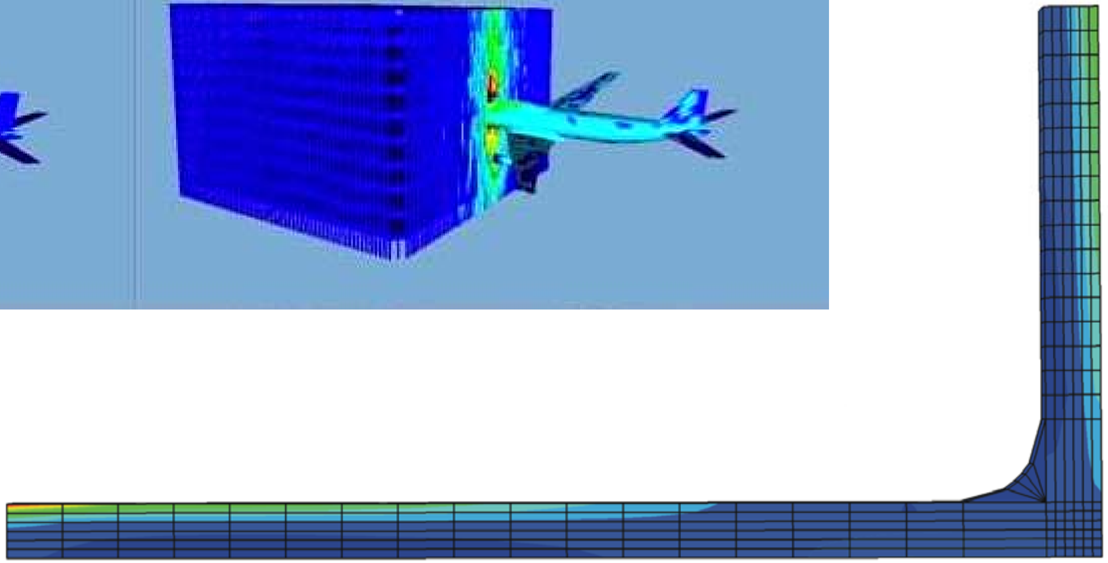
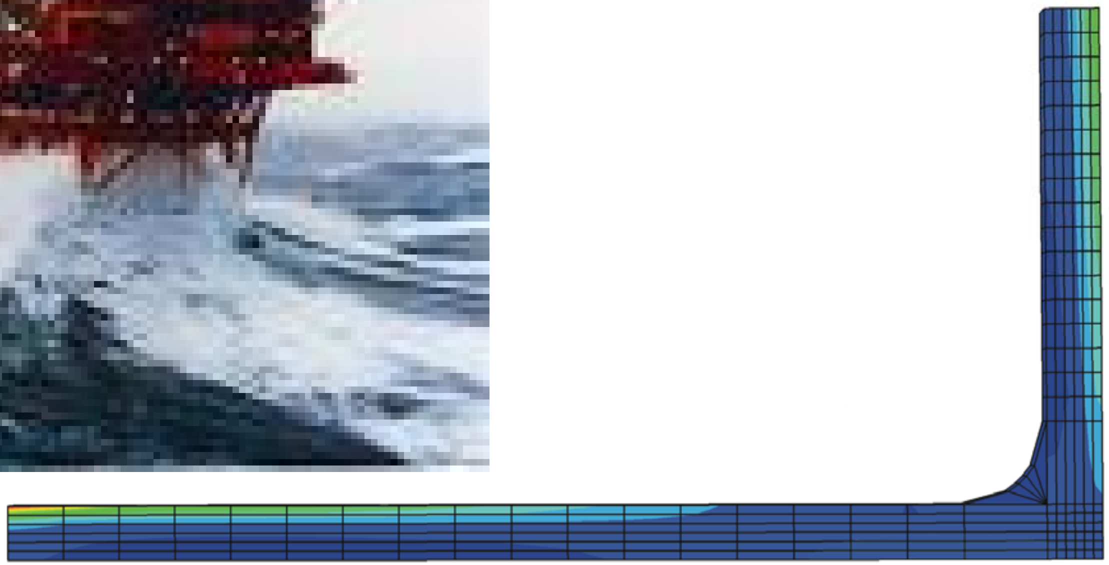




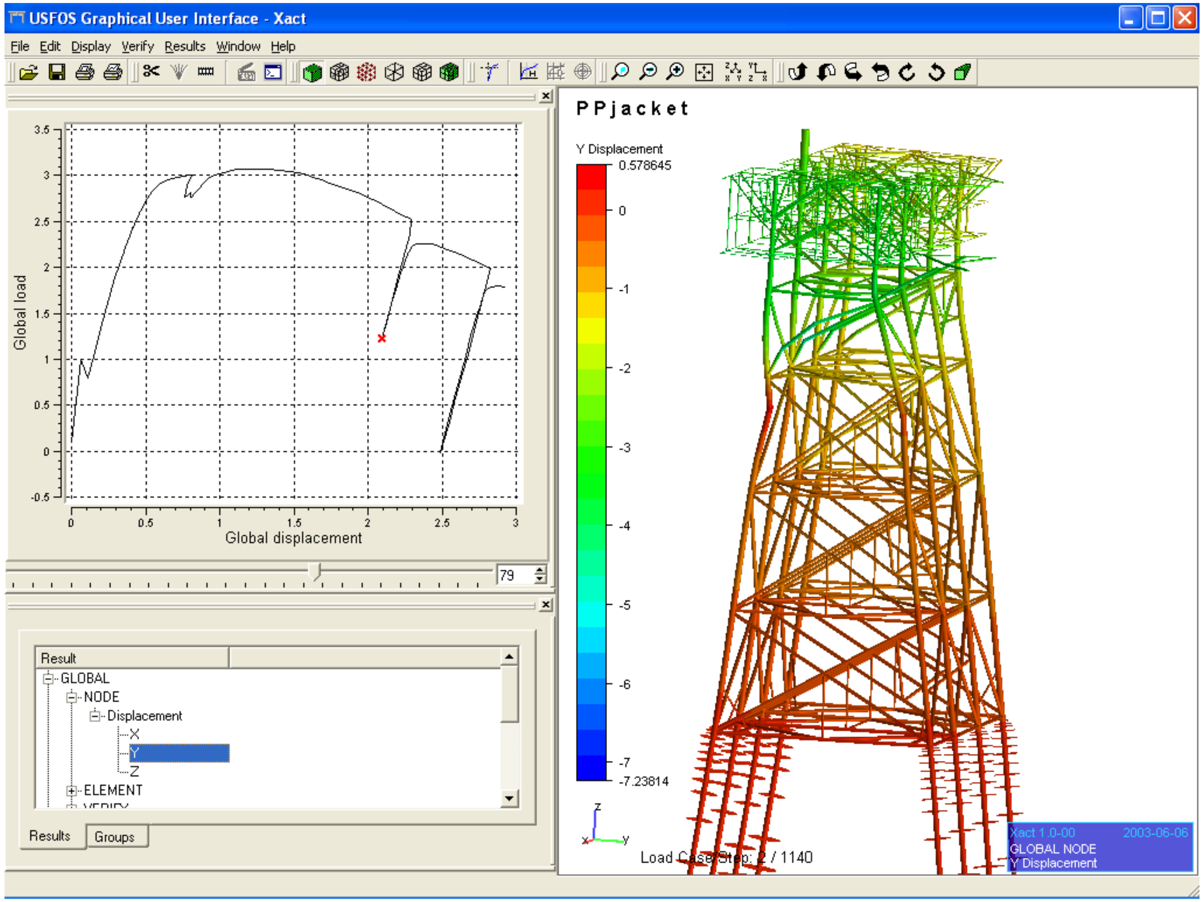
Motivation, overview and organization of the course

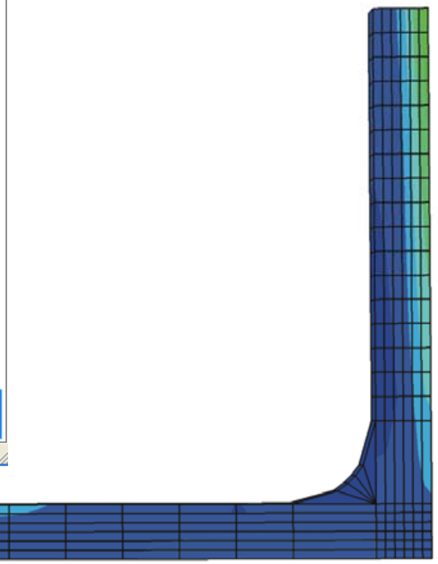
- **Analysis of ultimate collapse capacity of jacket structure**



Motivation, overview and organization of the course

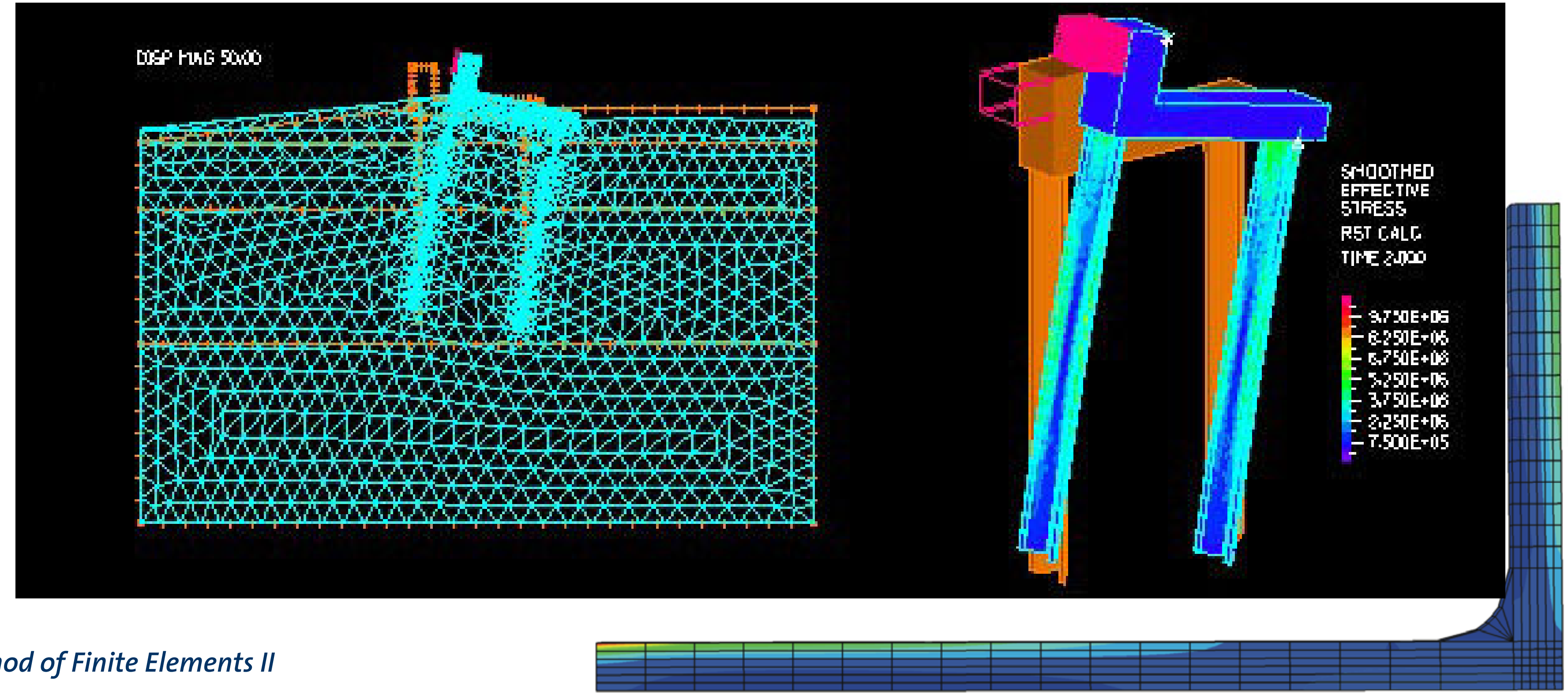
- **Analysis of ultimate collapse capacity of jacket structure**





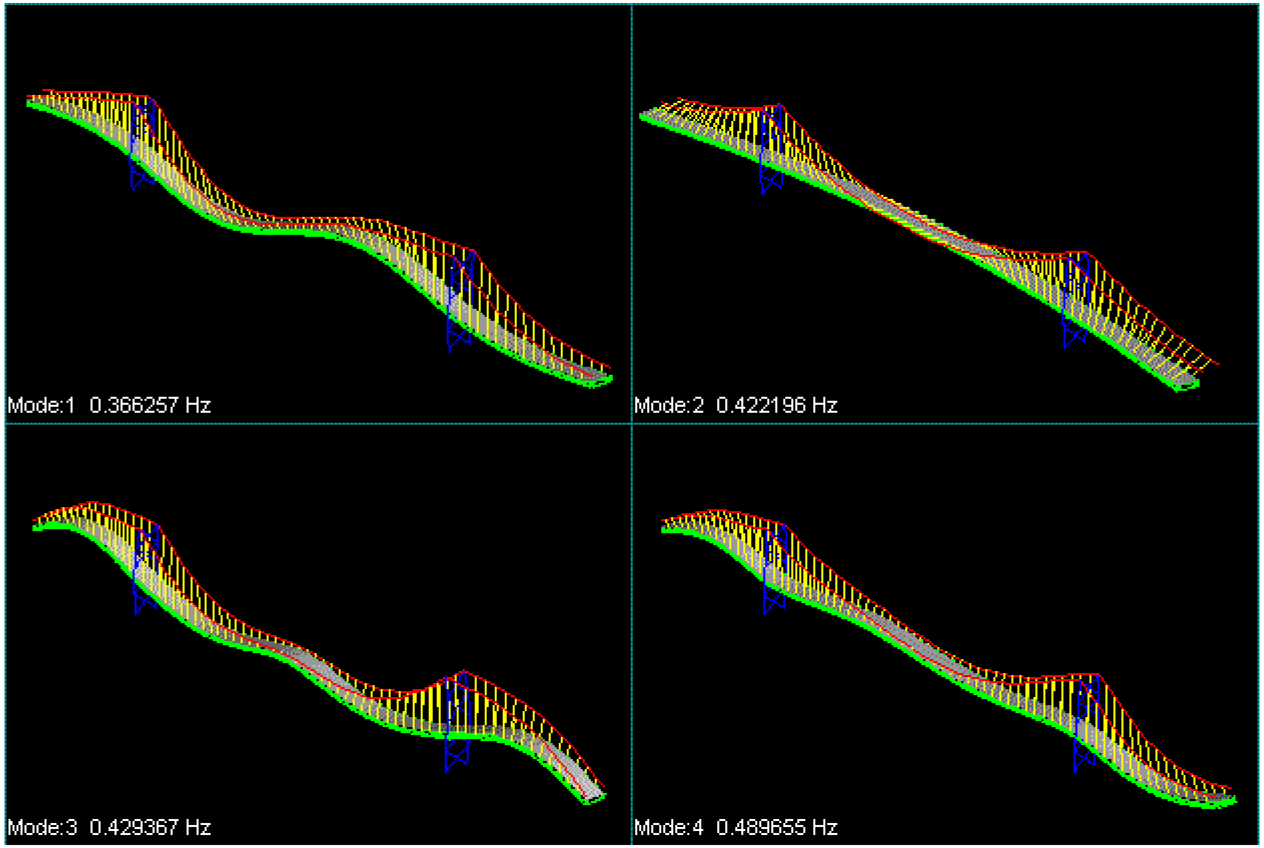
Motivation, overview and organization of the course

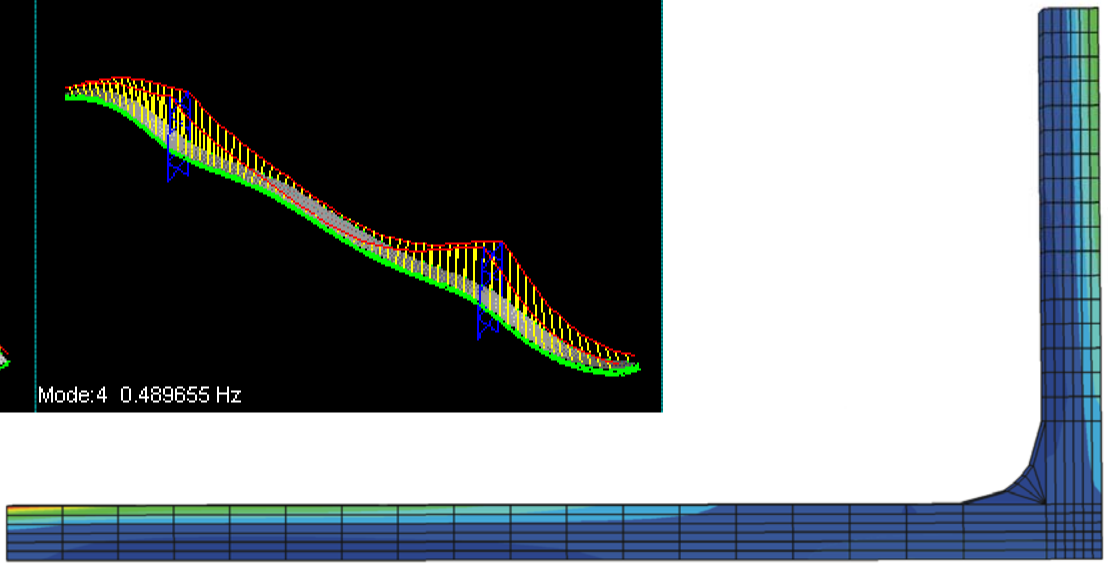
- **Analysis of soil performance**



Motivation, overview and organization of the course

- **Analysis of bridge response**





Motivation, overview and organization of the course

Steady state problems (Linear/Non-linear):

The response of the system does not change over time

$$\mathbf{KU} = \mathbf{R}$$

Propagation problems (Linear/Non-linear):

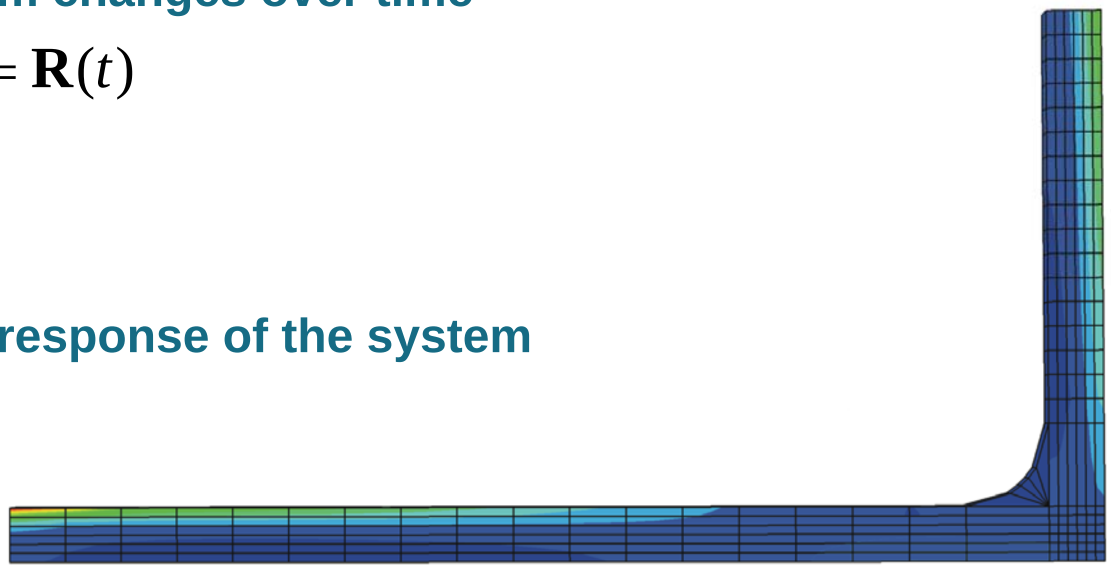
The response of the system changes over time

$$\mathbf{M\ddot{U}}(t) + \mathbf{C\dot{U}}(t) + \mathbf{KU}(t) = \mathbf{R}(t)$$

Eigenvalue problems:

No unique solution to the response of the system

$$\mathbf{Av} = \lambda \mathbf{Bv}$$



Motivation, overview and organization of the course

- **Organization**

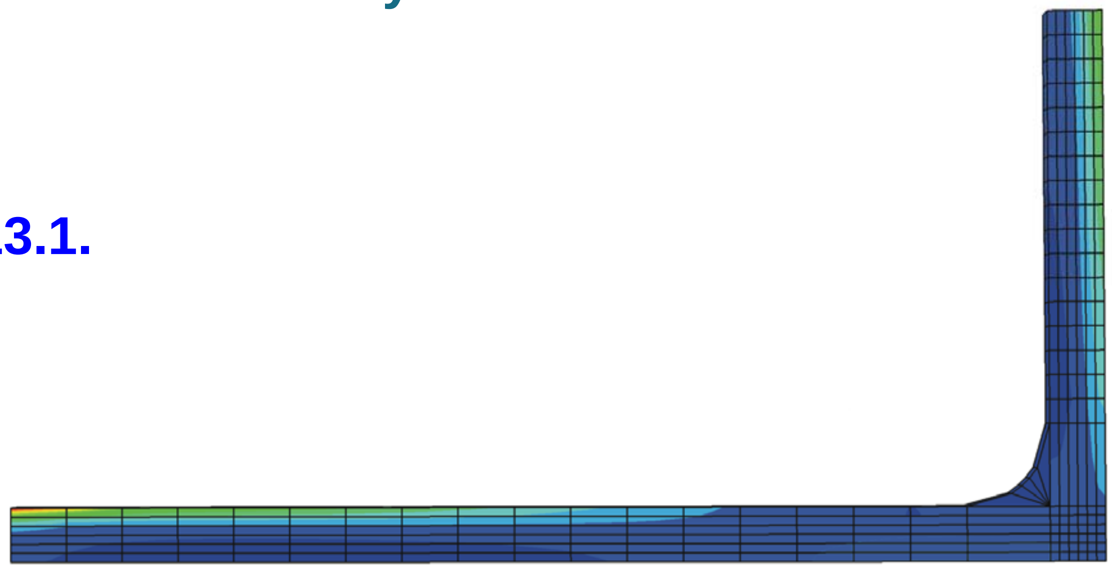
  The lectures will be given by:

  M. H. Faber

  Exercises will be organized/attended by:

  Jianjun Qin

  By appointment, HIL E13.1.



Motivation, overview and organization of the course

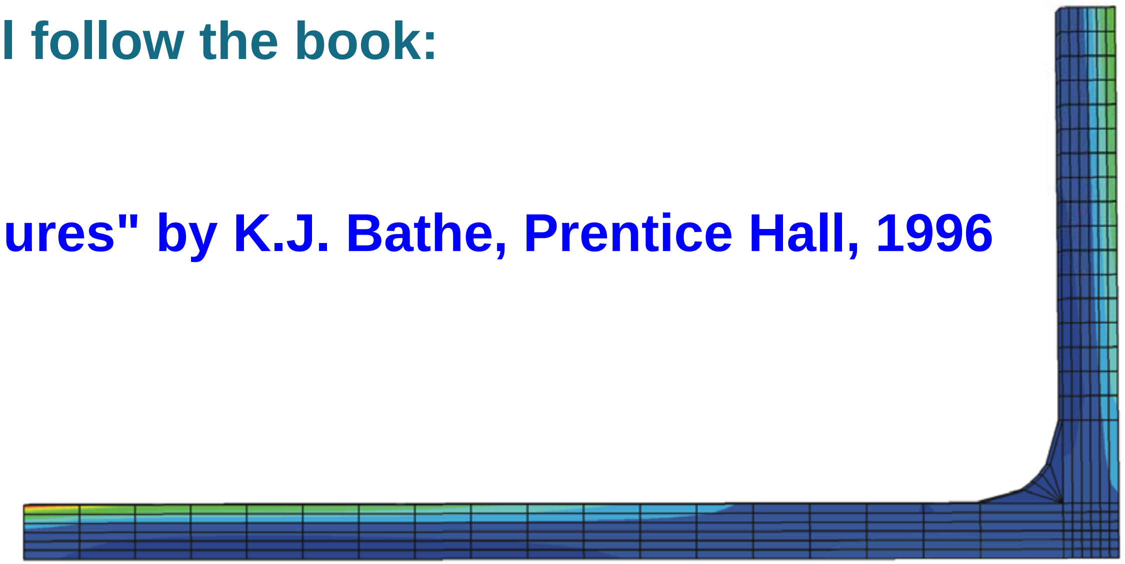
- **Organization**

  PowerPoint files with the presentations will be uploaded on our homepage one day in advance of the lectures

  http://www.ibk.ethz.ch/fa/education/FE_II

  The lecture as such will follow the book:

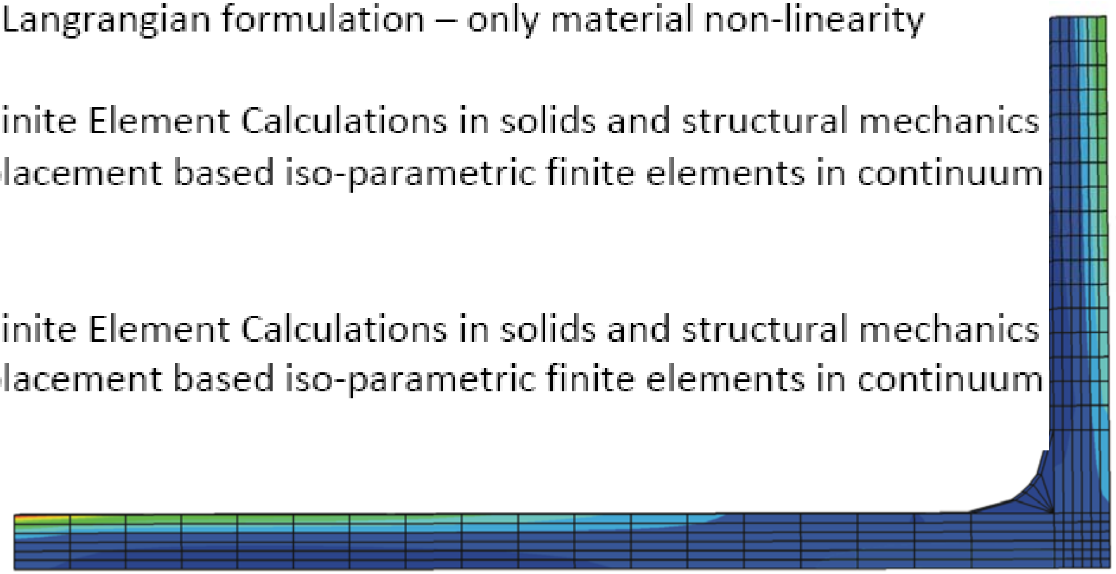
  "Finite Element Procedures" by K.J. Bathe, Prentice Hall, 1996



Motivation, overview and organization of the course

- ## Overview

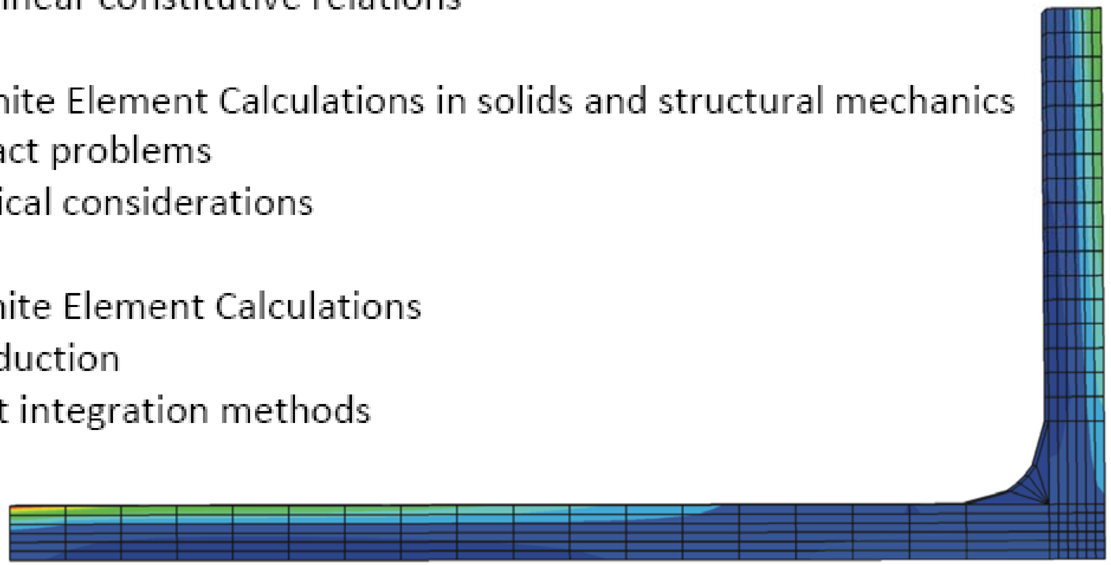
Date	Pages	Subject
18.09.2009	485-502	Non-linear Finite Element Calculations in solids and structural mechanics - Introduction to non-linear calculations - The incremental approach to continuum mechanics
25.09.2009	502-528	Non-linear Finite Element Calculations in solids and structural mechanics - Deformation gradients, strain and stress tensors - The Langrangian formulation – only material non-linearity
02.10.2009	538-548	Non-linear Finite Element Calculations in solids and structural mechanics - Displacement based iso-parametric finite elements in continuum mechanics
09.10.2009	548-560	Non-linear Finite Element Calculations in solids and structural mechanics - Displacement based iso-parametric finite elements in continuum mechanics



Motivation, overview and organization of the course

- **Overview**

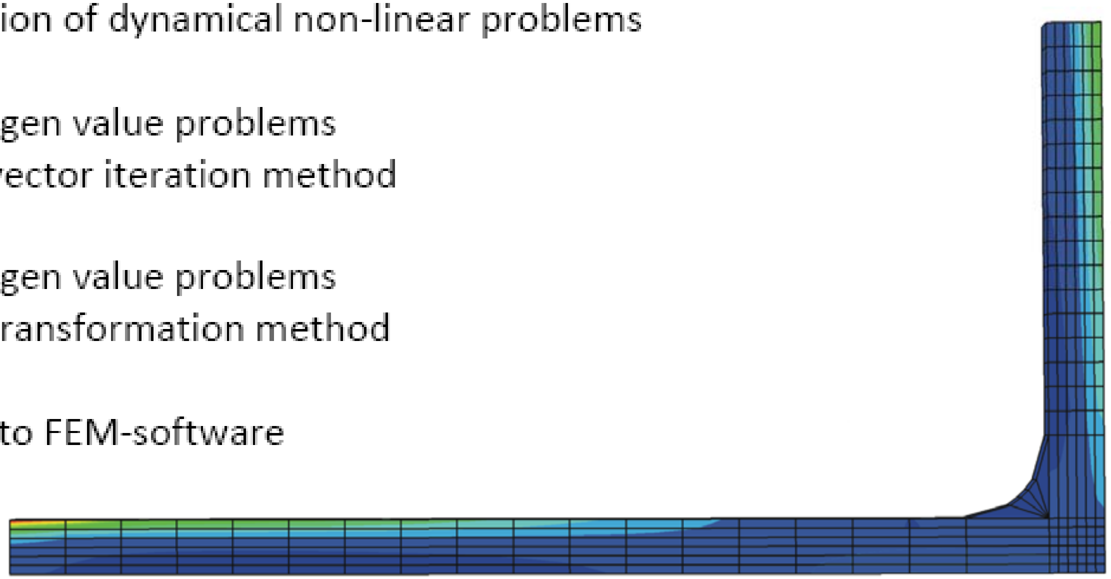
16.10.2009	561-578	Non-linear Finite Element Calculations in solids and structural mechanics
		- Total Langrangian formulation
		- Extended Langrangian formulation
		- Structural elements
23.10.2009	581-617	Non-linear Finite Element Calculations in solids and structural mechanics
		- Introduction to constitutive relations
		- Non-linear constitutive relations
30.10.2009	622-640	Non-linear Finite Element Calculations in solids and structural mechanics
		- Contact problems
		- Practical considerations
06.11.2009	768-784	Dynamical Finite Element Calculations
		- Introduction
		- Direct integration methods



Motivation, overview and organization of the course

- **Overview**

13.11.2009	785-800	Dynamical Finite Element Calculations - Mode superposition
20.11.2009	801-815	Dynamical Finite Element Calculations - Analysis of direct integration methods
27.11.2009	824-830	Dynamical Finite Element Calculations - Solution of dynamical non-linear problems
04.12.2009	887-910	Solution of Eigen value problems - The vector iteration method
11.12.2009	911-937	Solution of Eigen value problems - The transformation method
18.12.2009		Introduction to FEM-software



Introduction to non-linear analysis

- Previously we considered the solution of the following linear and static problem:

$$\mathbf{KU} = \mathbf{R}$$

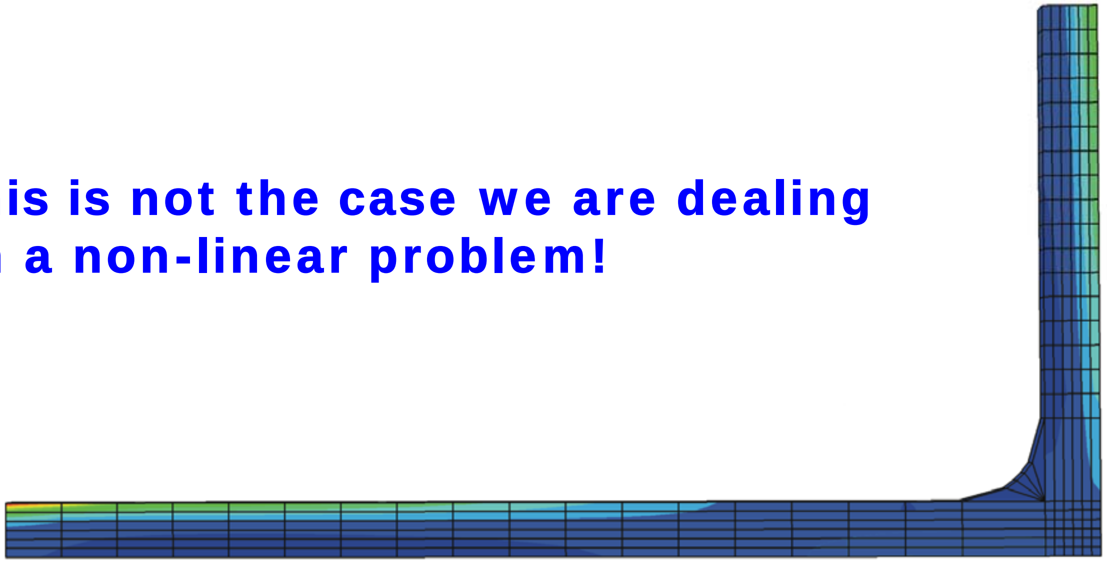
for these problems we have the convenient property of linearity, i.e:

$$\mathbf{KU} = \lambda\mathbf{R}, \quad \lambda = 1$$

$$\Downarrow$$

$$\mathbf{U}^* = \lambda\mathbf{U}, \quad \lambda \neq 1$$

If this is not the case we are dealing with a non-linear problem!



Introduction to non-linear analysis

- **Previously we considered the solution of the following linear and static problem:**

$$\mathbf{KU} = \mathbf{R}$$

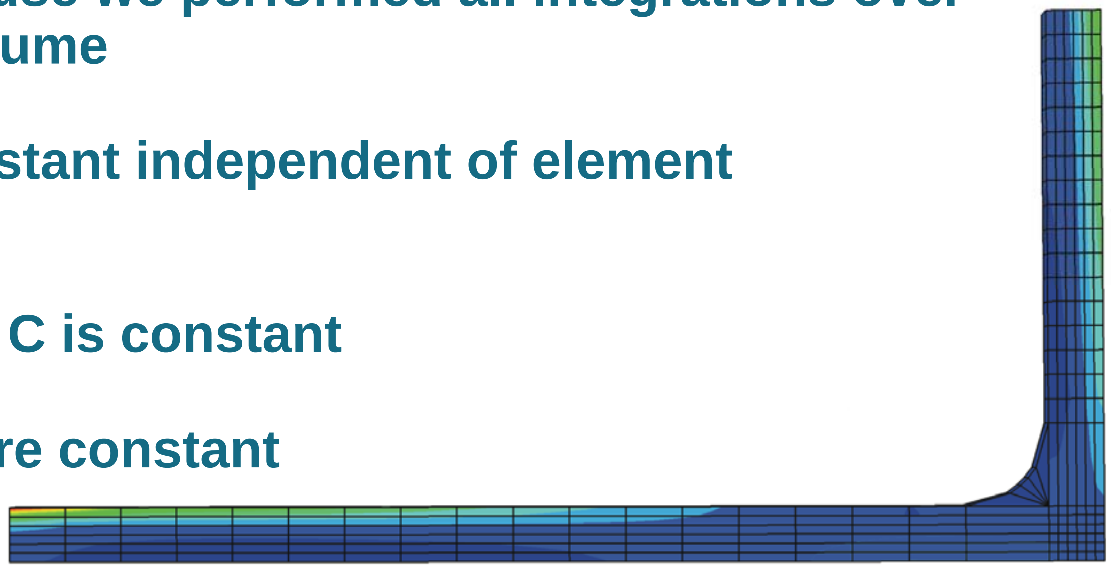
we assumed:

small displacements when developing the stiffness matrix K and the load vector R, because we performed all integrations over the original element volume

that the B matrix is constant independent of element displacements

the stress-strain matrix C is constant

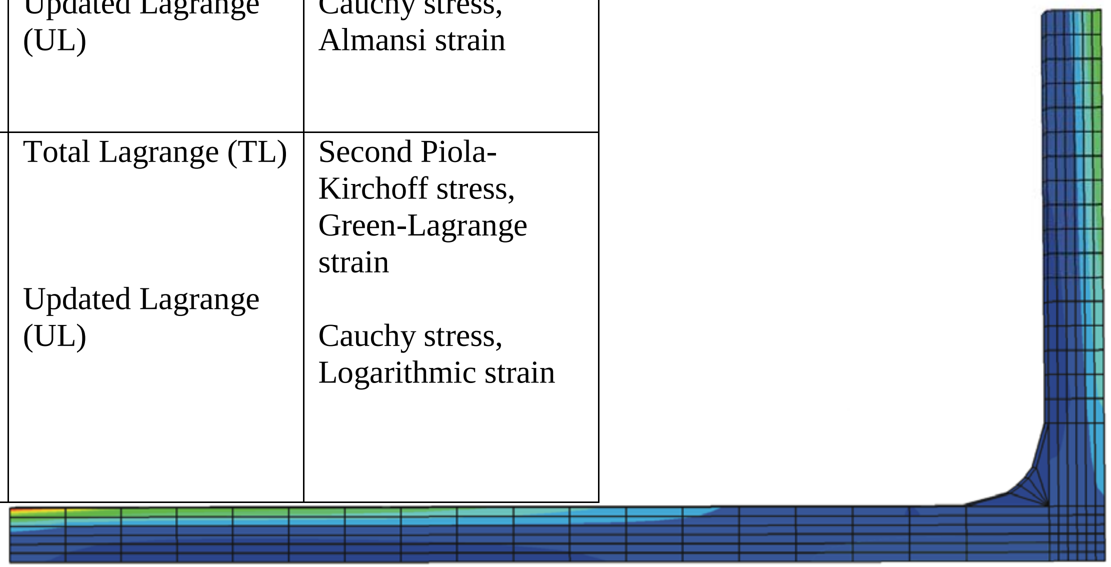
boundary constraints are constant



Introduction to non-linear analysis

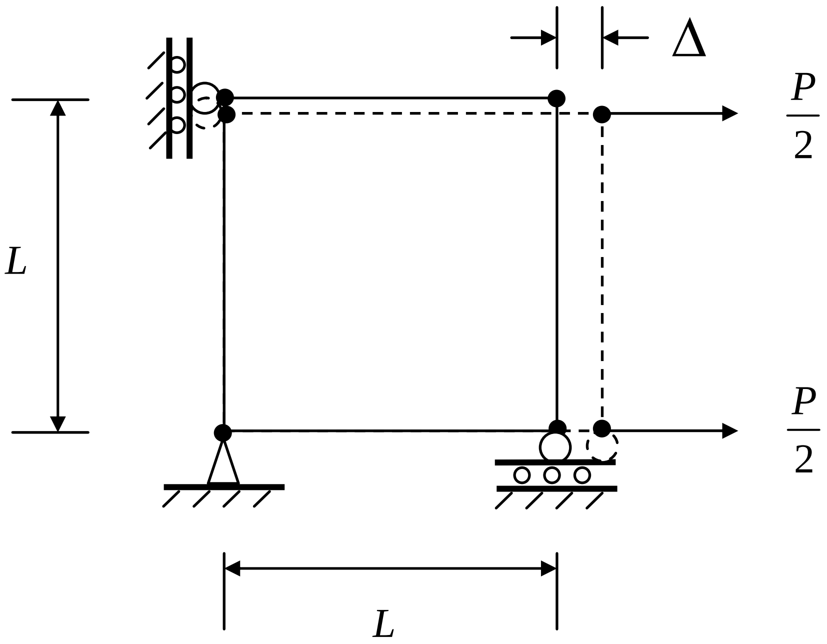
• Classification of non-linear analyses

Type of analysis	Description	Typical formulation used	Stress and strain measures used
Materially-nonlinear only	Infinitesimal displacements and strains; stress train relation is non-linear	Materially-nonlinear-only (MNO)	Engineering strain and stress
Large displacements, large rotations but small strains	Displacements and rotations of fibers are large; but fiber extensions and angle changes between fibers are small; stress strain relationship may be linear or non-linear	Total Lagrange (TL) Updated Lagrange (UL)	Second Piola-Kirchoff stress, Green-Lagrange strain Cauchy stress, Almansi strain
Large displacements, large rotations and large strains	Displacements and rotations of fibers are large; fiber extensions and angle changes between fibers may also be large; stress strain relationship may be linear or non-linear	Total Lagrange (TL) Updated Lagrange (UL)	Second Piola-Kirchoff stress, Green-Lagrange strain Cauchy stress, Logarithmic strain



Introduction to non-linear analysis

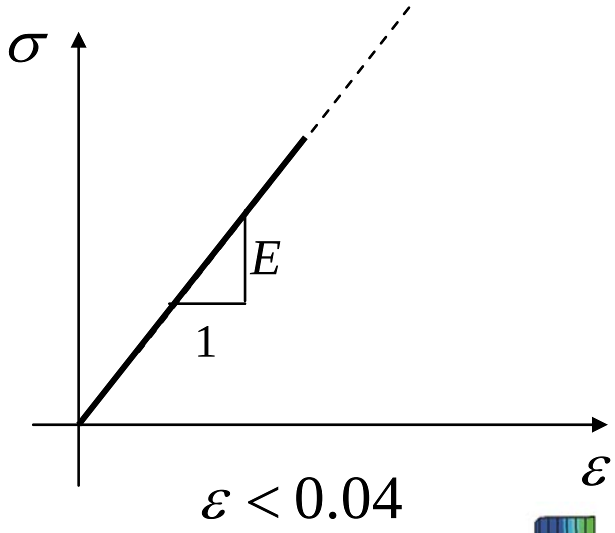
- ## Classification of non-linear analyses

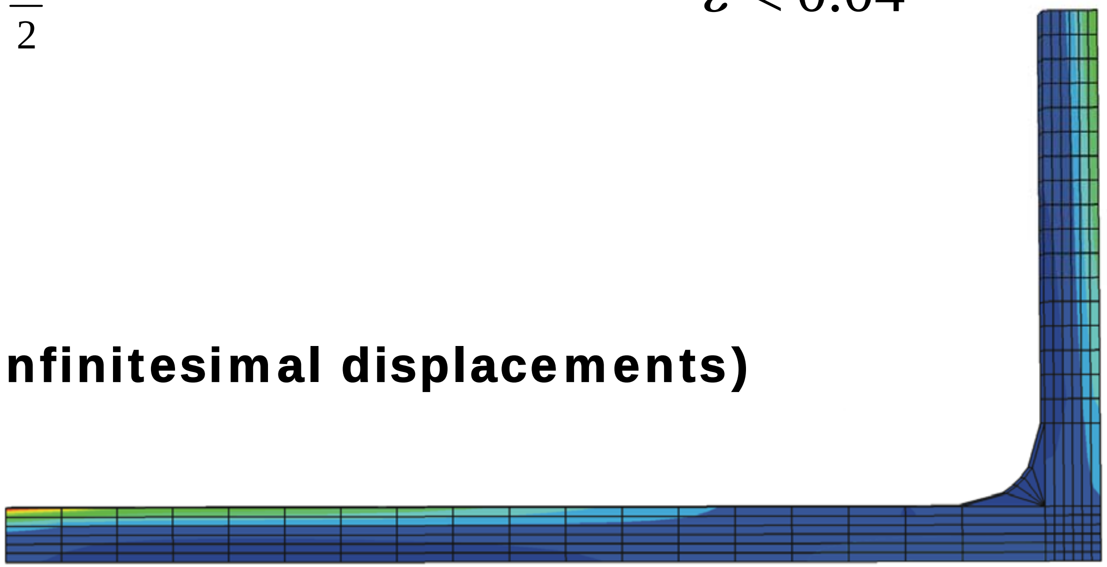


$$\sigma = P / A$$

$$\varepsilon = \sigma / E$$

$$\Delta = \varepsilon L$$

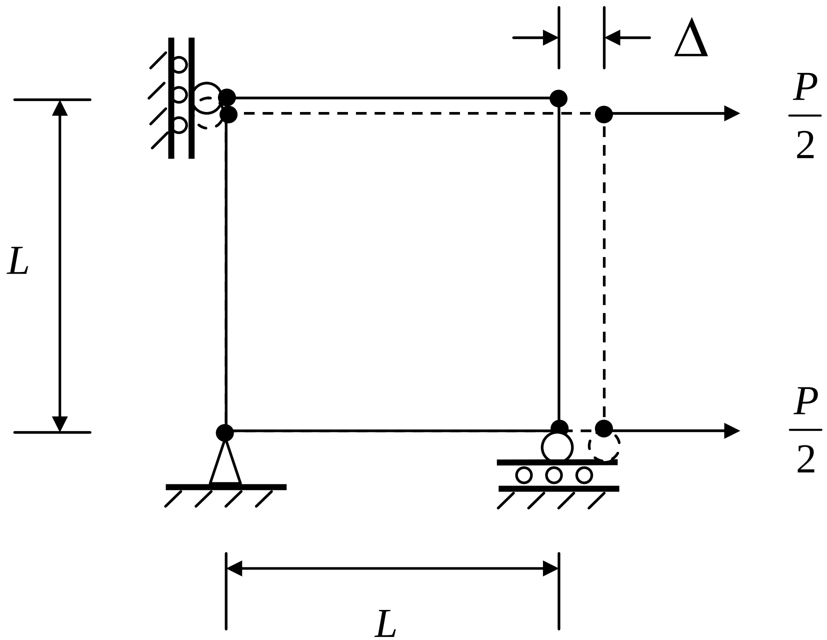




Linear elastic (infinitesimal displacements)

Introduction to non-linear analysis

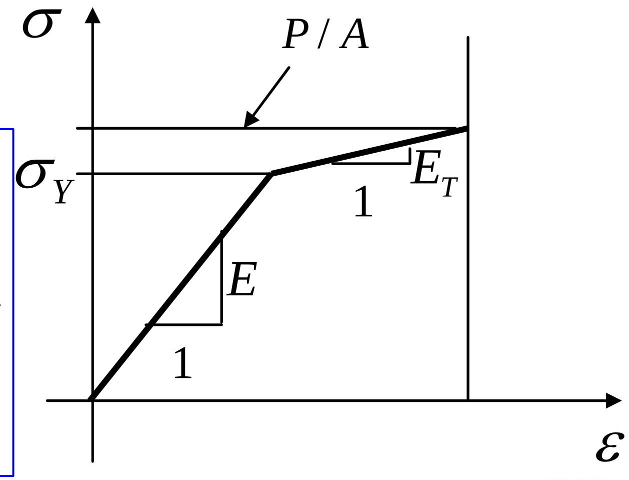
- ## Classification of non-linear analyses

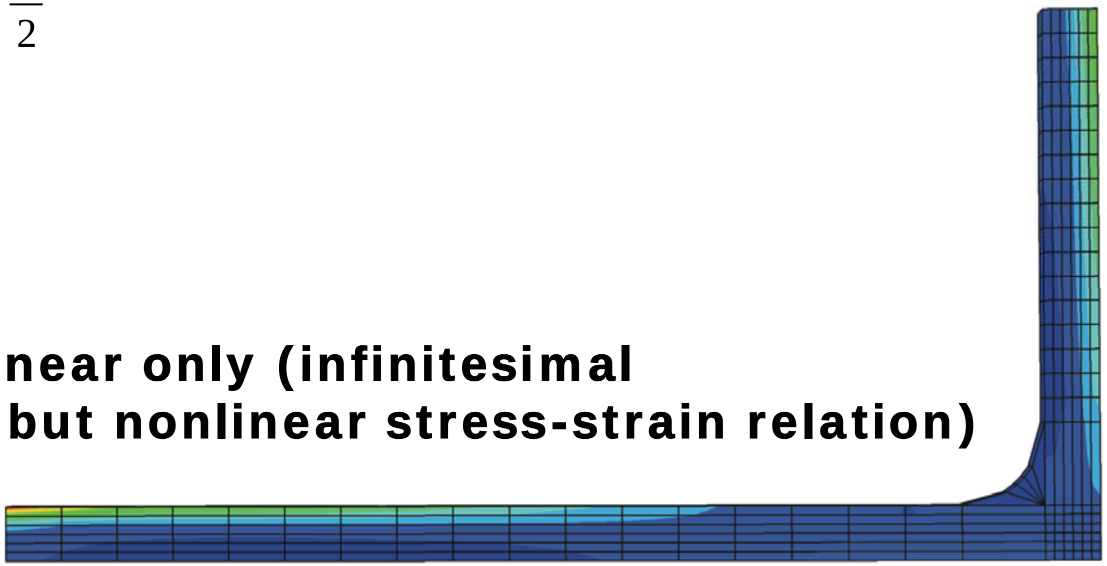


$$\sigma = P / A$$

$$\varepsilon = \frac{\sigma_Y}{E} + \frac{\sigma - \sigma_Y}{E_T}$$

$$\varepsilon < 0.04$$

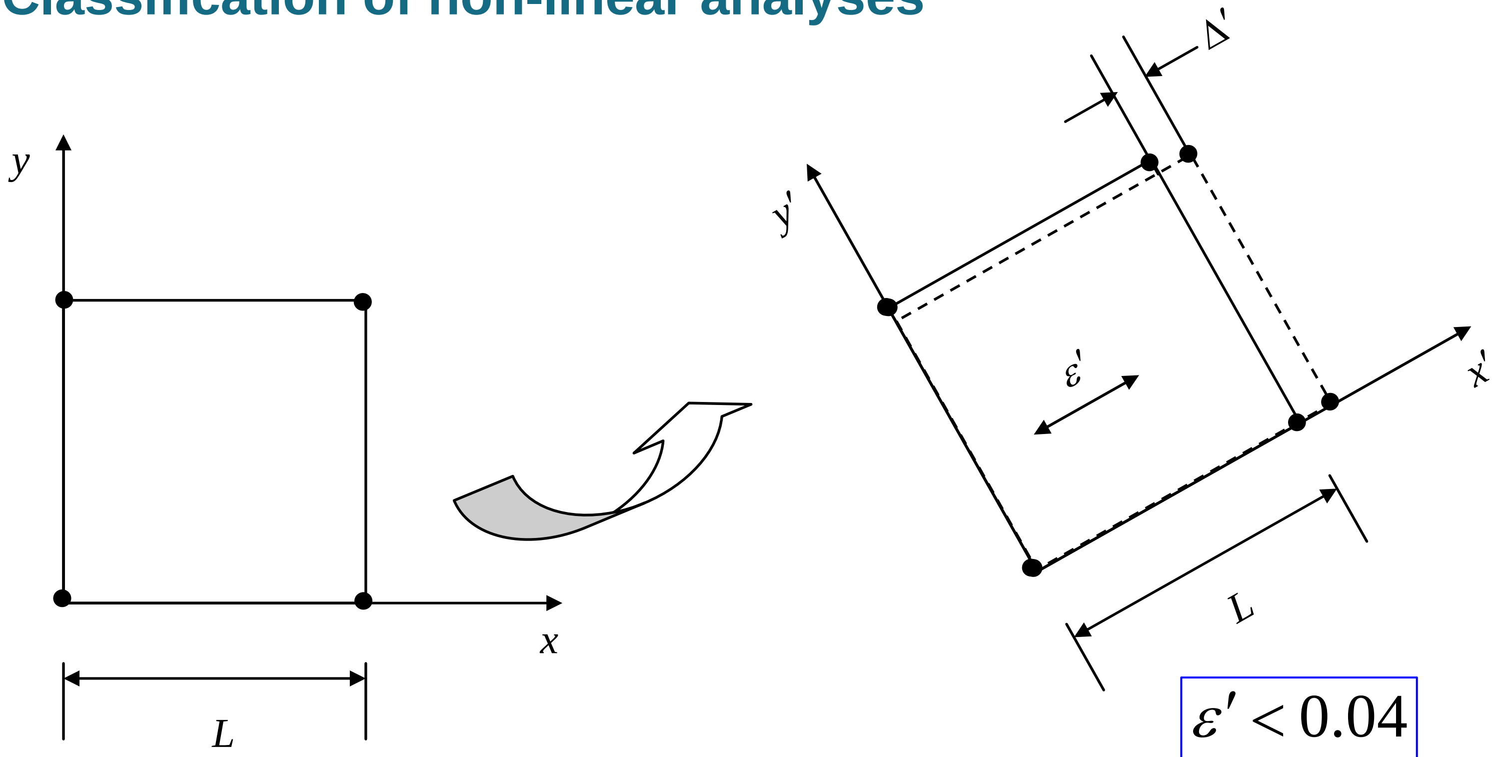




Materially nonlinear only (infinitesimal displacements, but nonlinear stress-strain relation)

Introduction to non-linear analysis

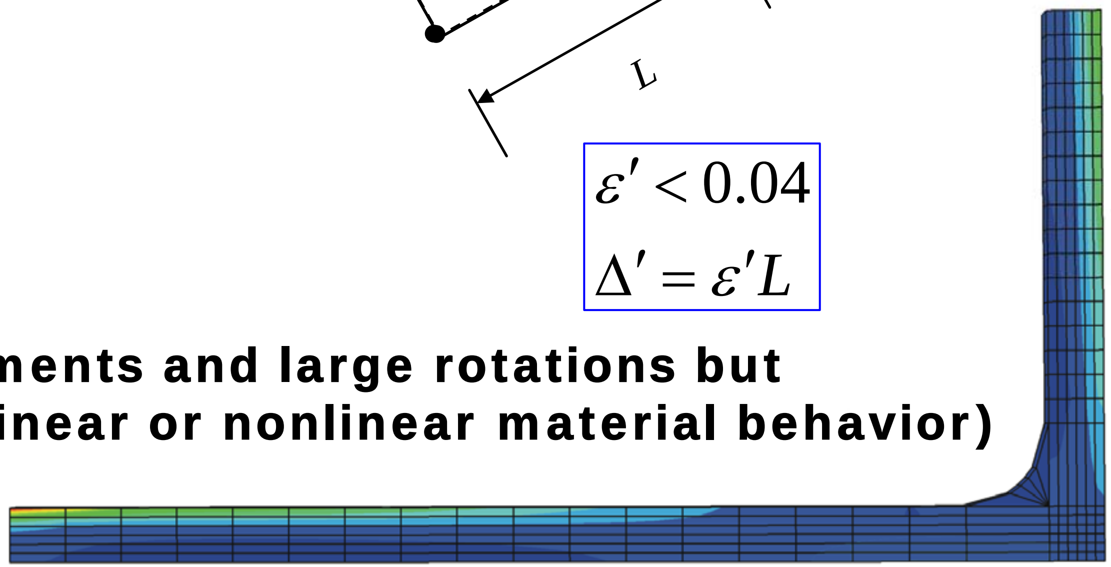
- **Classification of non-linear analyses**



$$\varepsilon' < 0.04$$

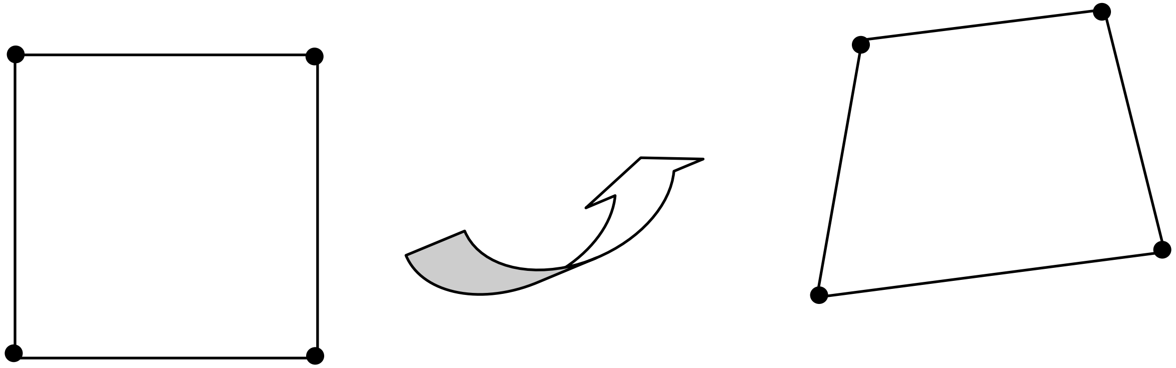
$$\Delta' = \varepsilon' L$$

Large displacements and large rotations but small strains (linear or nonlinear material behavior)

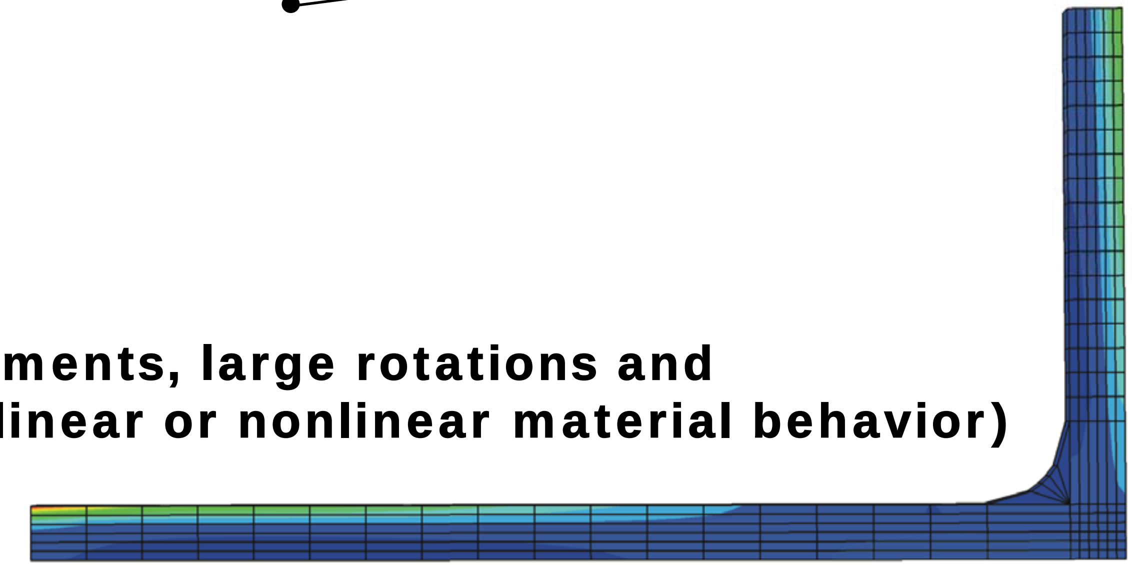


Introduction to non-linear analysis

- **Classification of non-linear analyses**

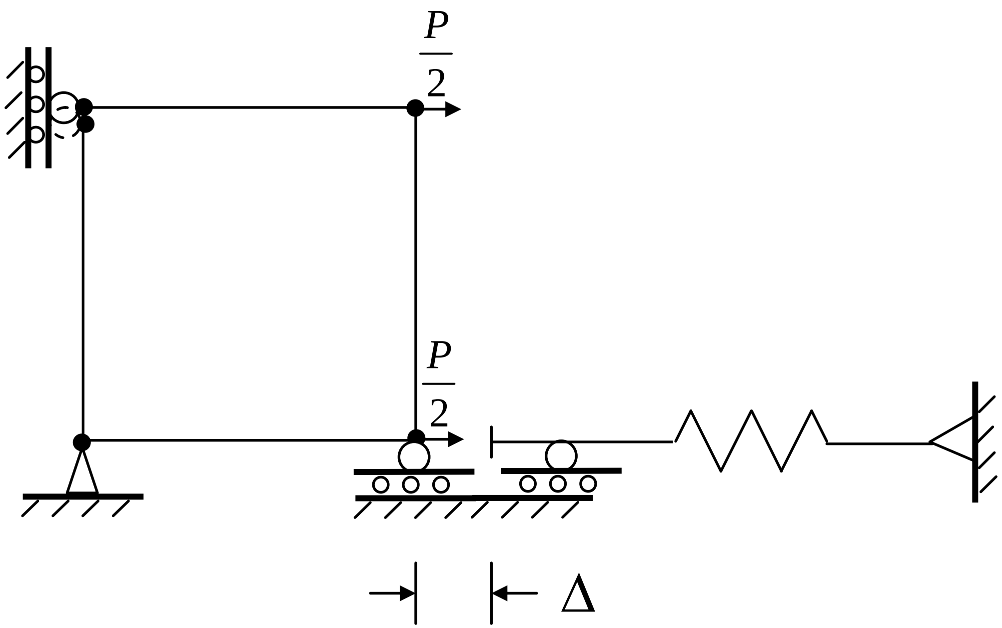


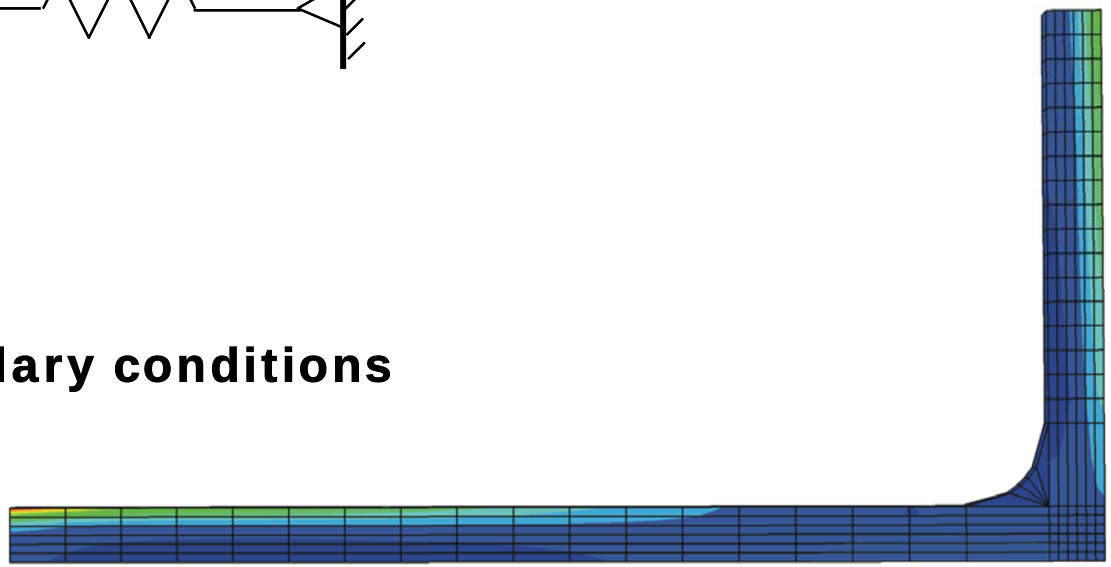
Large displacements, large rotations and large strains (linear or nonlinear material behavior)



Introduction to non-linear analysis

- Classification of non-linear analyses





Chang in boundary conditions

Introduction to non-linear analysis

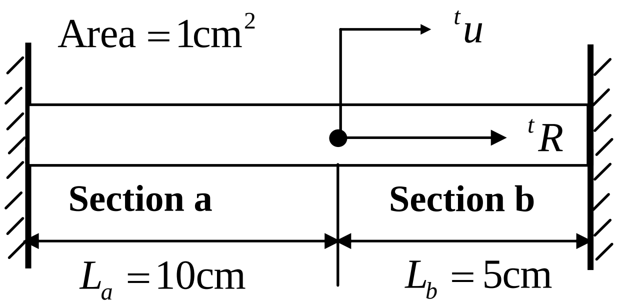
- **Example: Simple bar structure**

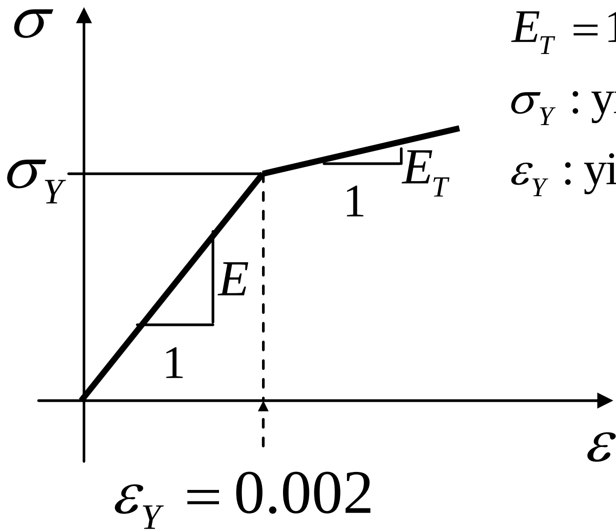
$E = 10^7 \ \mathrm{N/cm^2}$

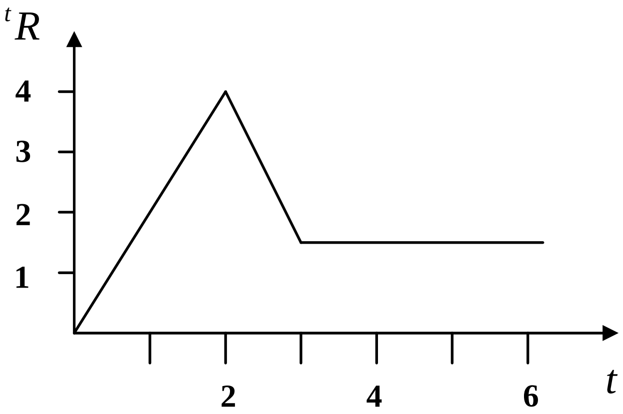
$E_T = 10^5 \ \mathrm{N/cm^2}$

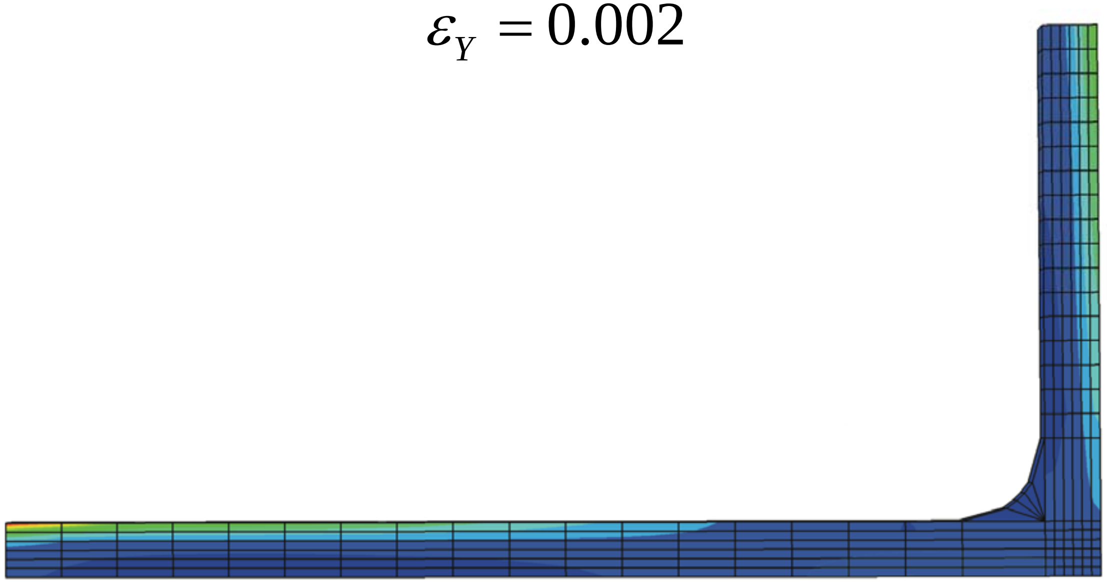
σ_Y : yield stress

ε_Y : yield strain



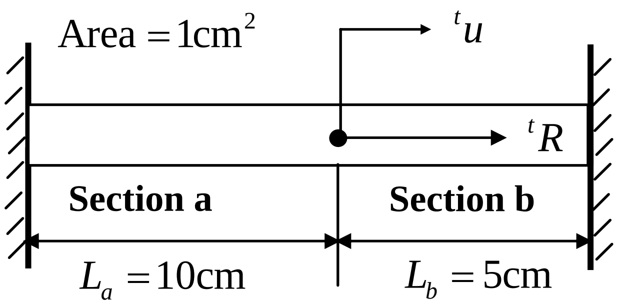






Introduction to non-linear analysis

- ## Example: Simple bar structure



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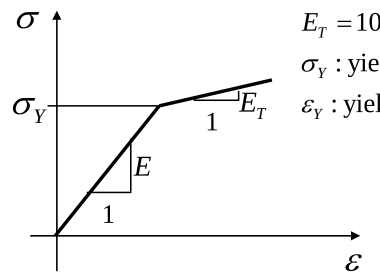
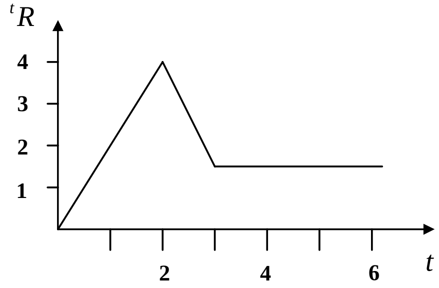


$\varepsilon = 0.002$



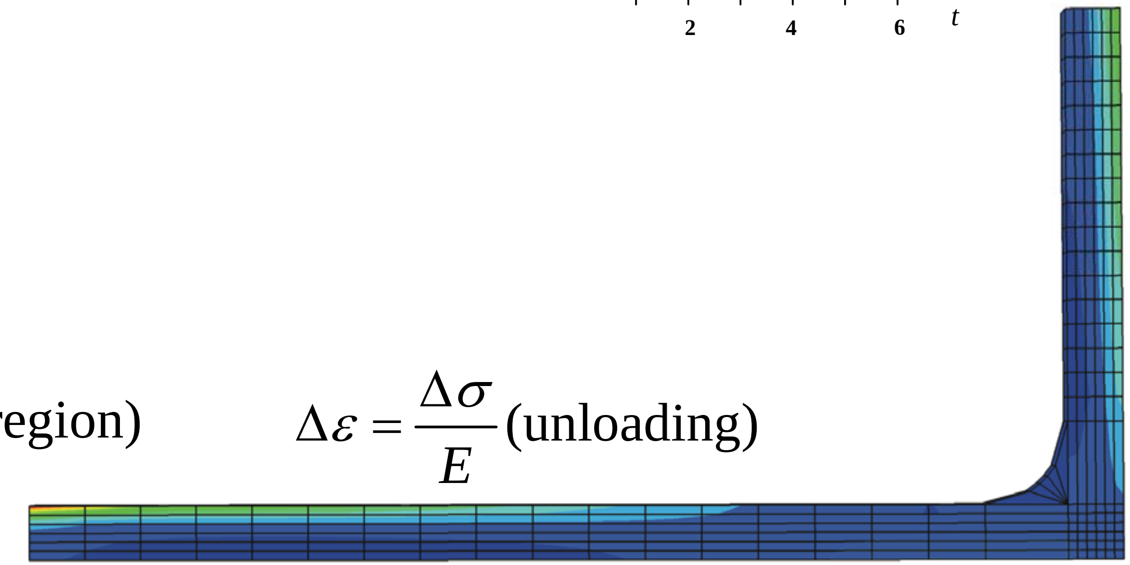
$$ {}^t\varepsilon_a = \frac{{}^t u}{L_a},\ {}^t\varepsilon_b = -\frac{{}^t u}{L_b} $$

$$ {}^tR + {}^t\sigma_b A = {}^t\sigma_a A $$

$$ {}^t\varepsilon = \frac{{}^t\sigma}{E}\ \text{(elastic region)} $$

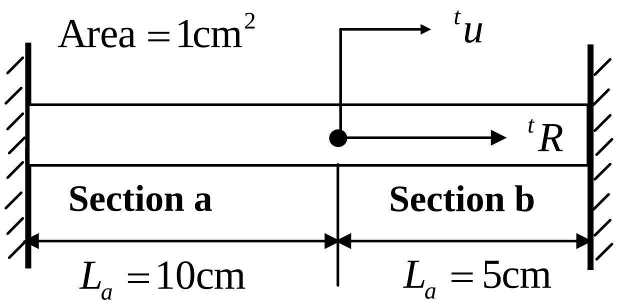
$$ {}^t\varepsilon = \varepsilon_Y + \frac{{}^t\sigma - \sigma_Y}{E_T}\ \text{(plastic region)} $$

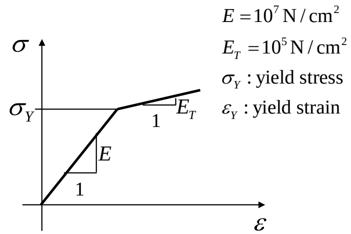
$$ \Delta\varepsilon = \frac{\Delta\sigma}{E}\ \text{(unloading)} $$

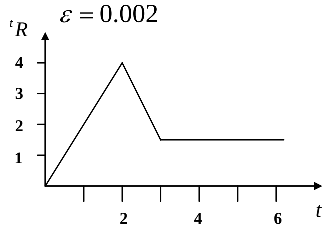


Introduction to non-linear analysis

- **Example: Simple bar structure**







$$ {}^t\varepsilon_a = \frac{{}^t u}{L_a}, \; {}^t\varepsilon_b = -\frac{{}^t u}{L_b} $$

$$ {}^tR + {}^t\sigma_b A = {}^t\sigma_a A $$

$$ {}^t\varepsilon = \frac{{}^t\sigma}{E} \quad \text{(elastic region)} $$

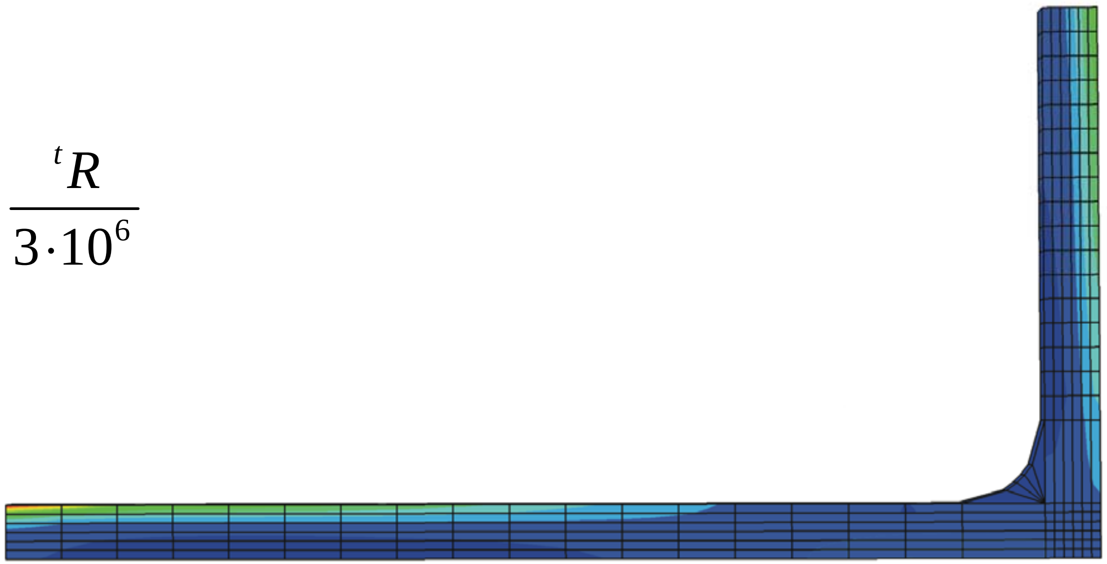
$$ {}^t\varepsilon = \varepsilon_Y + \frac{{}^t\sigma - \sigma_Y}{E_T} \quad \text{(plastic region)} $$

$$ \Delta\varepsilon = \frac{\Delta\sigma}{E} \text{(unloading)} $$

Both sections elastic

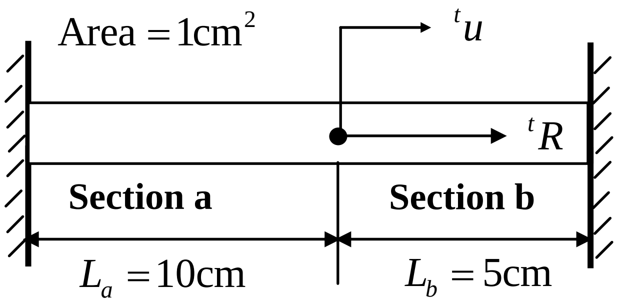
$$ {}^tR = EA\,{}^tu\left(\frac{1}{L_a} + \frac{1}{L_b}\right) \Rightarrow {}^tu = \frac{{}^tR}{3\cdot 10^6} $$

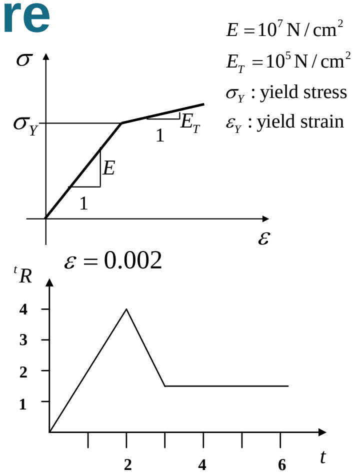
$$ \sigma_a = \frac{{}^tR}{3A}, \sigma_b = -\frac{2}{3}\frac{{}^tR}{A} $$



Introduction to non-linear analysis

• Example: Simple bar structure





$$ {}^t\varepsilon_a = \frac{{}^tu}{L_a},\ {}^t\varepsilon_b = -\frac{{}^tu}{L_b} $$

$$ {}^tR + {}^t\sigma_b A = {}^t\sigma_a A $$

$$ {}^t\varepsilon = \frac{{}^t\sigma}{E} \quad \text{(elastic region)} $$

$$ {}^t\varepsilon = \varepsilon_Y + \frac{{}^t\sigma - \sigma_Y}{E_T} \quad \text{(plastic region)} $$

$$ \Delta\varepsilon = \frac{\Delta\sigma}{E} \text{(unloading)} $$

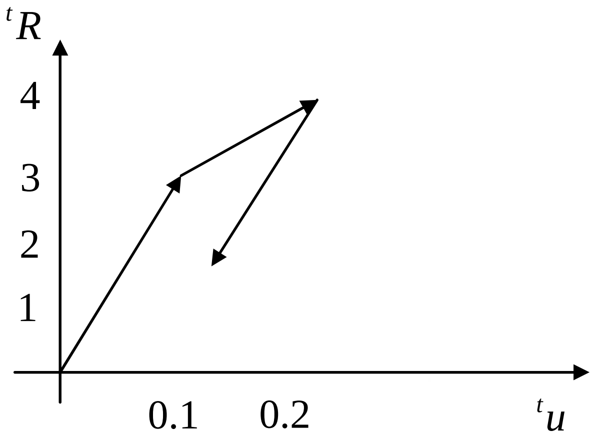
Section a is elastic while section b is plastic

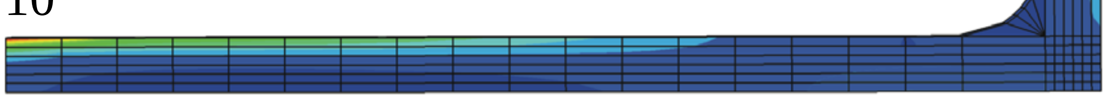
section b will be plastic when $ {}^{t^*}R = \frac{3}{2}\sigma_Y A $

$$ \sigma_a = E\frac{{}^tu}{L_a},\ \sigma_b = -E_T\left(\frac{{}^tu}{L_b} - \varepsilon_Y\right) - \sigma_Y $$

$$ {}^tR = \frac{EA\,{}^tu}{L_a} + \frac{E_T A\,{}^tu}{L_b} - E_T\varepsilon_Y A + \sigma_Y A \Rightarrow $$

$$ {}^tu = \frac{{}^tR/A + E_T\varepsilon_Y - \sigma_Y}{E/L_a + E/L_b} = \frac{{}^tR}{1.02\cdot 10^6} - 1.9412\cdot 10^{-2} $$





Introduction to non-linear analysis

- **What did we learn from the example?**

The basic problem in general nonlinear analysis is to find a state of equilibrium between externally applied loads and element nodal forces

$$ {}^{t}\mathbf{R} - {}^{t}\mathbf{F} = 0 $$

$$ {}^{t}\mathbf{R} = {}^{t}\mathbf{R}_{B} + {}^{t}\mathbf{R}_{S} + {}^{t}\mathbf{R}_{C} $$

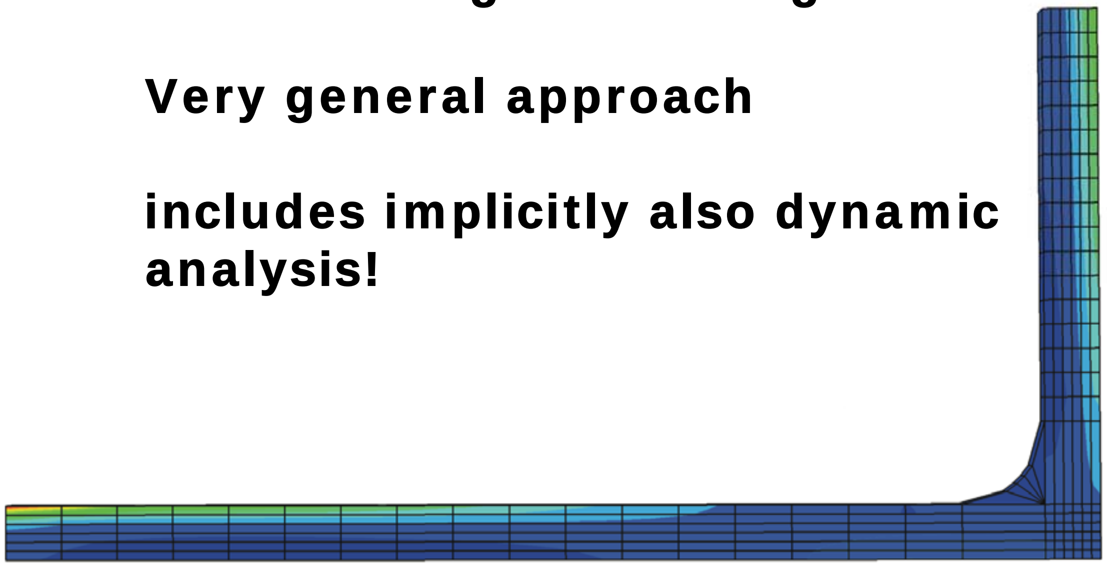
$$ {}^{t}\mathbf{F} = {}^{t}\mathbf{R}_{I} $$

$$ {}^{t}\mathbf{F} = \sum_{m} \int_{{}^{t}V^{(m)}} {}^{t}\mathbf{B}^{(m)T}\, {}^{t}\tau^{(m)}\, {}^{t}dV^{(m)} $$

We must achieve equilibrium for all time steps when incrementing the loading

Very general approach

includes implicitly also dynamic analysis!



Introduction to non-linear analysis

- **The basic approach in incremental analysis is**

$$^{t+\Delta t}\mathbf{R} - {}^{t+\Delta t}\mathbf{F} = 0$$

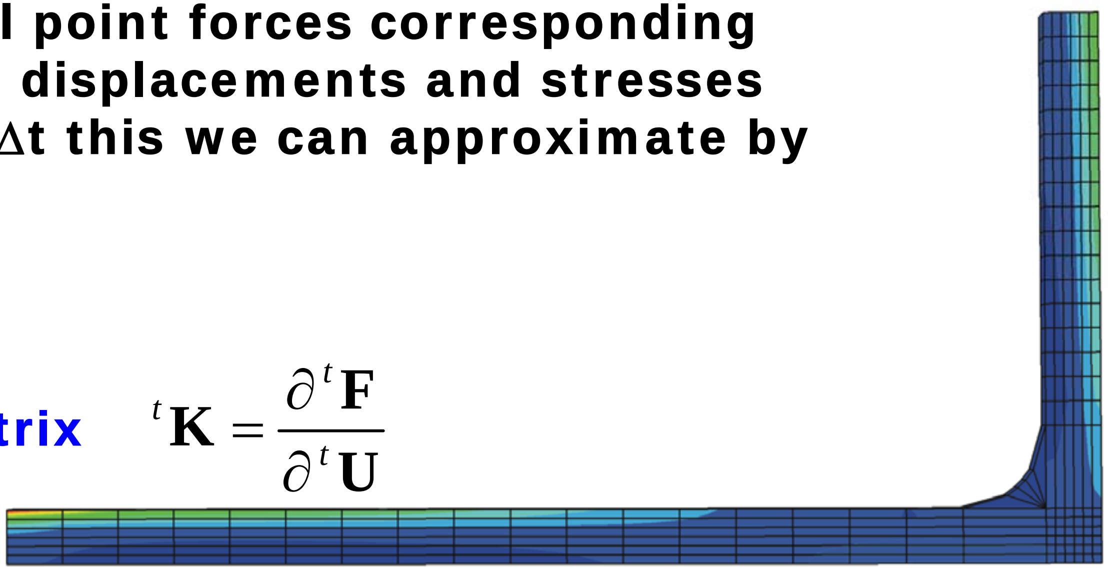
assuming that $^{t+\Delta t}\mathbf{R}$ is independent of the deformations we have

$$^{t+\Delta t}\mathbf{F} = {}^{t}\mathbf{F} + \mathbf{F}$$

We know the solution ^{t}F at time *t* and F is the increment in the nodal point forces corresponding to an increment in the displacements and stresses from time t to time t+Δt this we can approximate by

$$\mathbf{F} = {}^{t}\mathbf{K}\mathbf{U}$$

Tangent stiffness matrix $\quad {}^{t}\mathbf{K} = \dfrac{\partial\, {}^{t}\mathbf{F}}{\partial\, {}^{t}\mathbf{U}}$



Introduction to non-linear analysis

- The basic approach in incremental analysis is

We may now substitute the tangent stiffness matrix into the equlibrium relation

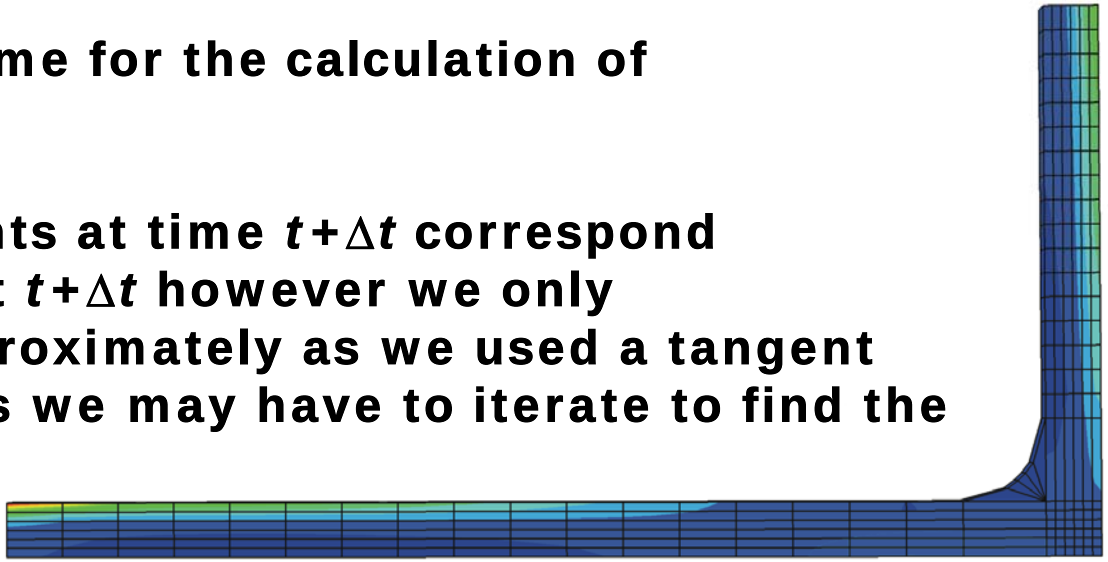
$$^{t}\mathbf{KU} = {}^{t+\Delta t}\mathbf{R} - {}^{t}\mathbf{F}$$

$$\Downarrow$$

$$^{t+\Delta t}\mathbf{U} = {}^{t}\mathbf{U} + \mathbf{U}$$

which gives us a scheme for the calculation of the displacements

the exact displacements at time *t*+Δ*t* correspond to the applied loads at *t*+Δ*t* however we only determined these approximately as we used a tangent stiffness matrix – thus we may have to iterate to find the solution



Introduction to non-linear analysis

- **The basic approach in incremental analysis is**

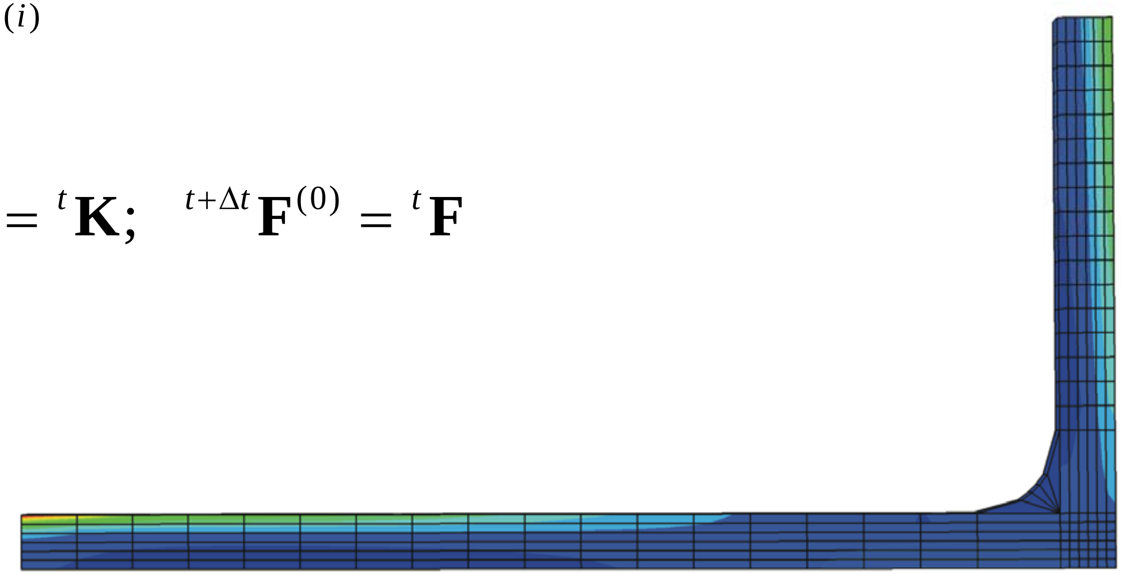
We may use the Newton-Raphson iteration scheme to find the equlibrium within each load increment

$$^{t+\Delta t}\mathbf{K}^{(i-1)}\Delta\mathbf{U}^{(i)} = {}^{t+\Delta t}\mathbf{R} - {}^{t+\Delta t}\mathbf{F}^{(i-1)}$$ **(out of balance load vector)**

$$^{t+\Delta t}\mathbf{U}^{(i)} = {}^{t+\Delta t}\mathbf{U}^{(i-1)} + \Delta\mathbf{U}^{(i)}$$

with initial conditions

$$^{t+\Delta t}\mathbf{U}^{(0)} = {}^{t}\mathbf{U}; \quad {}^{t+\Delta t}\mathbf{K}^{(0)} = {}^{t}\mathbf{K}; \quad {}^{t+\Delta t}\mathbf{F}^{(0)} = {}^{t}\mathbf{F}$$



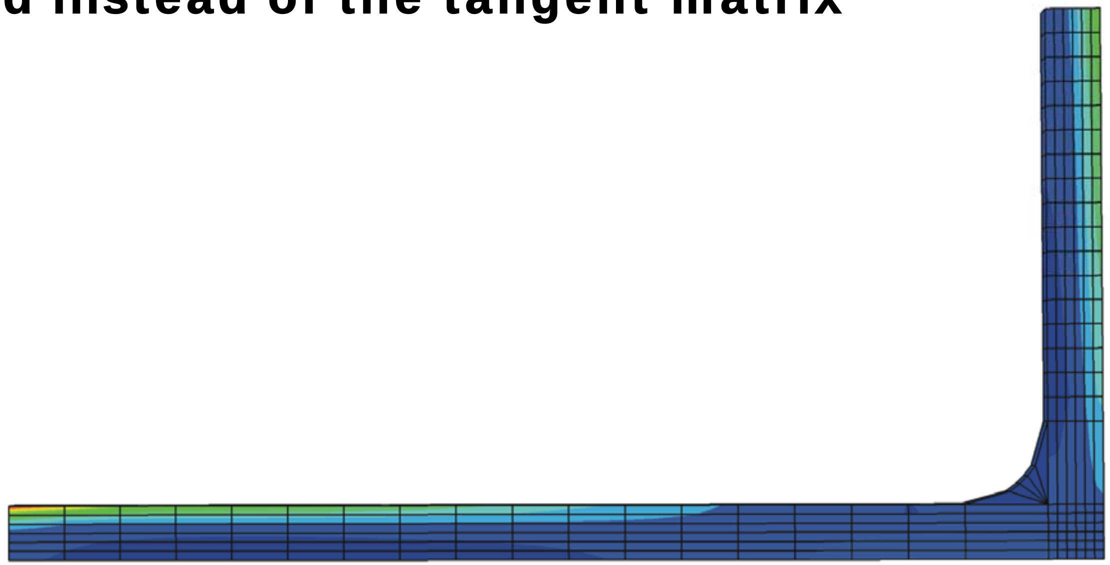
Introduction to non-linear analysis

- **The basic approach in incremental analysis is**

It may be expensive to calculate the tangent stiffness matrix and,

in the Modified Newton-Raphson iteration scheme it is thus only calculated in the beginning of each new load step

in the quasi-Newton iteration schemes the secant stiffness matrix is used instead of the tangent matrix



Introduction to non-linear analysis

- We look at the example again – simple bar (two load steps)

$$({}^{t}K_a + {}^{t}K_b)\Delta u^{(i)} = {}^{t+\Delta t}R - ({}^{t+\Delta t}F_a^{(i-1)} - {}^{t+\Delta t}F_b^{(i-1)})$$

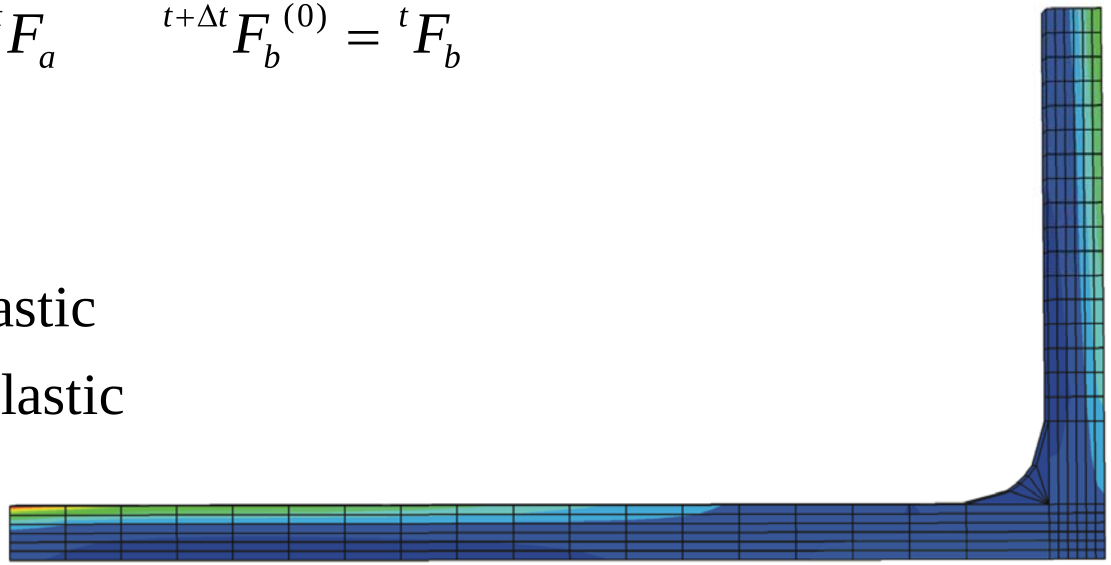
$${}^{t+\Delta t}u^{(i)} = {}^{t+\Delta t}u^{(i-1)} + \Delta u^{(i)}$$

with initial conditions

$${}^{t+\Delta t}u^{(0)} = {}^{t}u; \quad {}^{t+\Delta t}F_a^{(0)} = {}^{t}F_a \qquad {}^{t+\Delta t}F_b^{(0)} = {}^{t}F_b$$

$${}^{t}K_a = \frac{{}^{t}CA}{L_a}; \quad {}^{t}K_b = \frac{{}^{t}CA}{L_b}$$

$${}^{t}C \begin{cases} = E & \text{if section is elastic} \\ = E_T & \text{if section is plastic} \end{cases}$$



Introduction to non-linear analysis

- **We look at the example again – simple bar**

Load step 1: $t = 1$:

$$({}^{0}K_a + {}^{0}K_b)\Delta u^{(1)} = {}^{1}R - {}^{1}F_a^{(0)} - {}^{1}F_b^{(0)}$$

$$\Downarrow$$

$$\Delta u^{(1)} = \frac{2\times 10^{4}}{10^{7}(\frac{1}{10}+\frac{1}{5})} = 6.6667\times 10^{-3}$$

Iteration 1: $(i = 1)$

$${}^{1}u^{(1)} = {}^{1}u^{(0)} + \Delta u^{(1)} = 6.6667\times 10^{-3}$$

$${}^{1}\varepsilon_a^{(1)} = \frac{{}^{1}u^{(1)}}{L_a} = 6.6667\times 10^{-4} < \varepsilon_Y \text{ (elastic section!)}$$

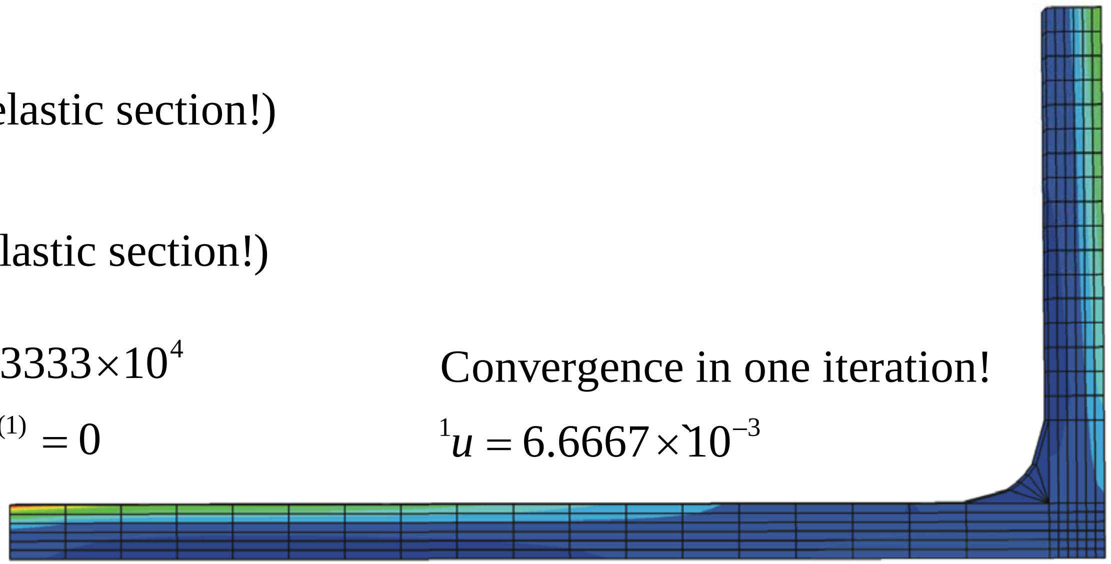
$${}^{1}\varepsilon_b^{(1)} = \frac{{}^{1}u^{(1)}}{L_b} = 1.3333\times 10^{-3} < \varepsilon_Y \text{ (elastic section!)}$$

$${}^{1}F_a^{(1)} = 6.6667\times 10^{3}; \qquad {}^{1}F_b^{(1)} = 1.3333\times 10^{4}$$

$$({}^{0}K_a + {}^{0}K_b)\Delta u^{(2)} = {}^{1}R - {}^{1}F_a^{(1)} - {}^{1}F_b^{(1)} = 0$$

Convergence in one iteration!

$${}^{1}u = 6.6667\times 10^{-3}$$



Introduction to non-linear analysis

- **We look at the example again – simple bar**

Load step 2: $t = 2$:

$$({}^1K_a + {}^1K_b)\Delta u^{(1)} = {}^2R - {}^2F_a^{(0)} - {}^2F_b^{(0)}$$

$$\Downarrow$$

$$\Delta u^{(1)} = \frac{(4\times10^4)-(6.6667\times10^3)-(1.333\times10^4)}{10^7(\frac{1}{10}+\frac{1}{5})} = 6.6667\times10^{-3}$$

Iteration 1: $(i=1)$

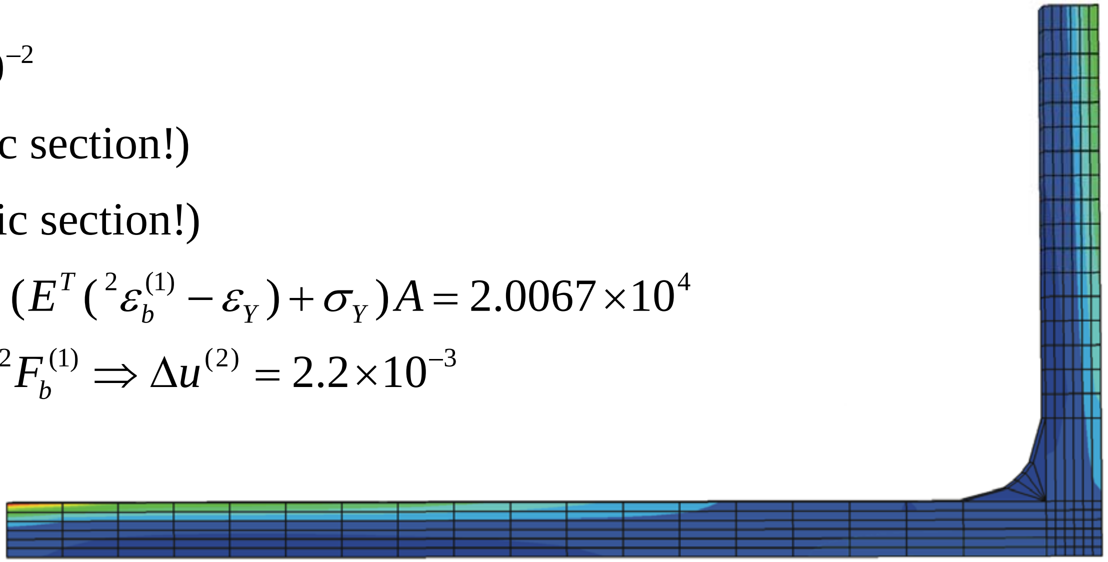
$${}^2u^{(1)} = {}^2u^{(0)} + \Delta u^{(1)} = 1.3333\times10^{-2}$$

$${}^2\varepsilon_a^{(1)} = 1.3333\times10^{-3} < \varepsilon_Y \text{ (elastic section!)}$$

$${}^2\varepsilon_b^{(1)} = 2.6667\times10^{-3} > \varepsilon_Y \text{ (plastic section!)}$$

$${}^1F_a^{(1)} = 1.3333\times10^4; \qquad {}^1F_b^{(1)} = (E^T({}^2\varepsilon_b^{(1)} - \varepsilon_Y) + \sigma_Y)A = 2.0067\times10^4$$

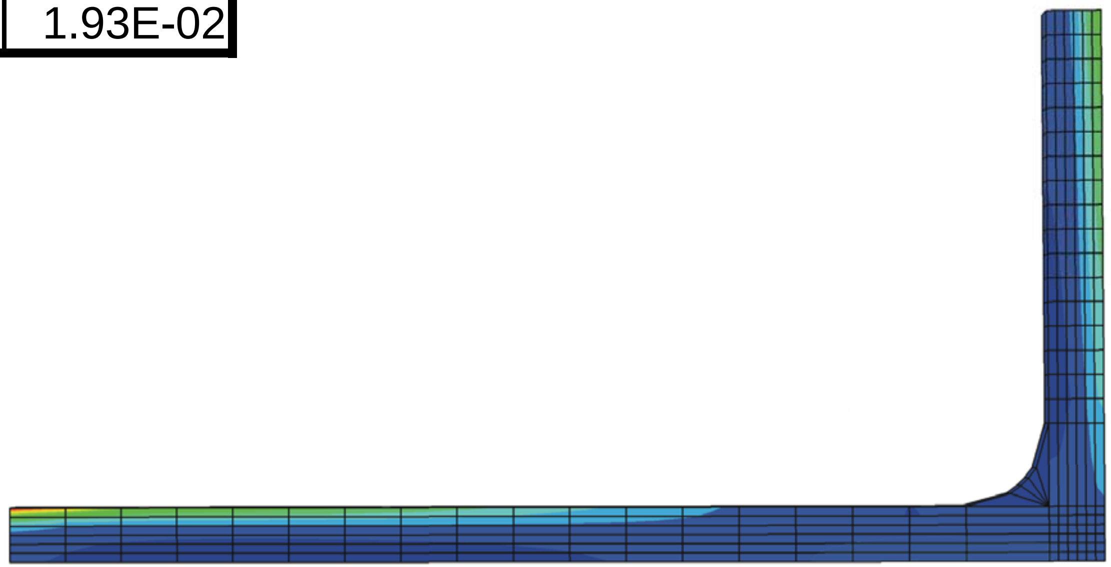
$$({}^1K_a + {}^1K_b)\Delta u^{(2)} = {}^2R - {}^2F_a^{(1)} - {}^2F_b^{(1)} \Rightarrow \Delta u^{(2)} = 2.2\times10^{-3}$$



Introduction to non-linear analysis

- **We look at the example again – simple bar**

i	$\Delta u^{(i)}$	${}^{2}u^{(i)}$
2	1.45E-03	1.55E-02
3	1.45E-03	1.70E-02
4	9.58E-04	1.79E-02
5	6.32E-04	1.86E-02
6	4.17E-04	1.90E-02
7	2.76E-04	1.93E-02



The continuum mechanics incremental equations

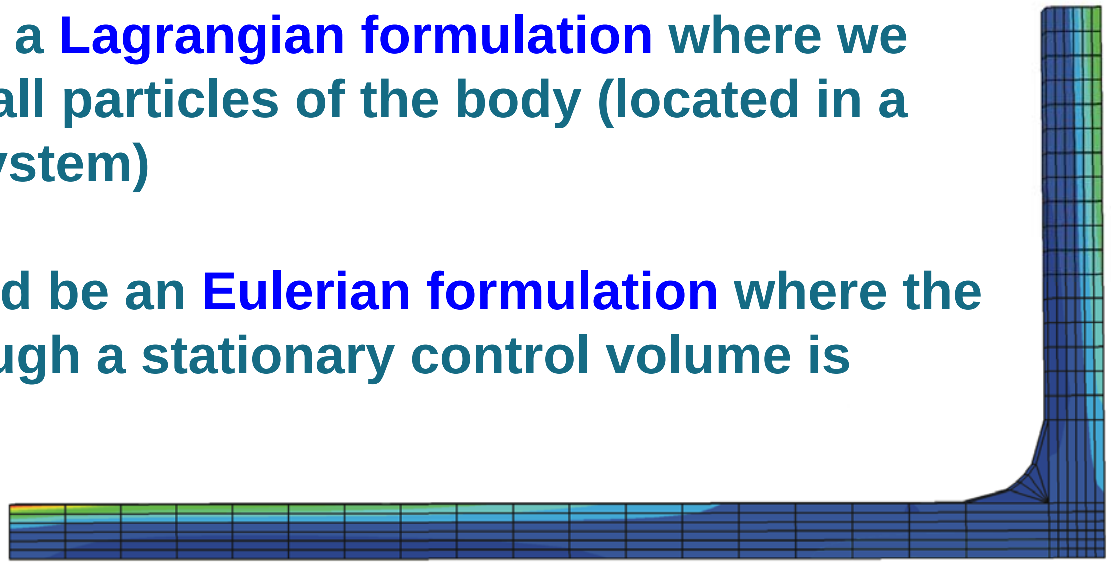
- **The basic problem:**

  We want to establish the solution using an incremental formulation

  The equilibrium must be established for the considered body in its current configuration

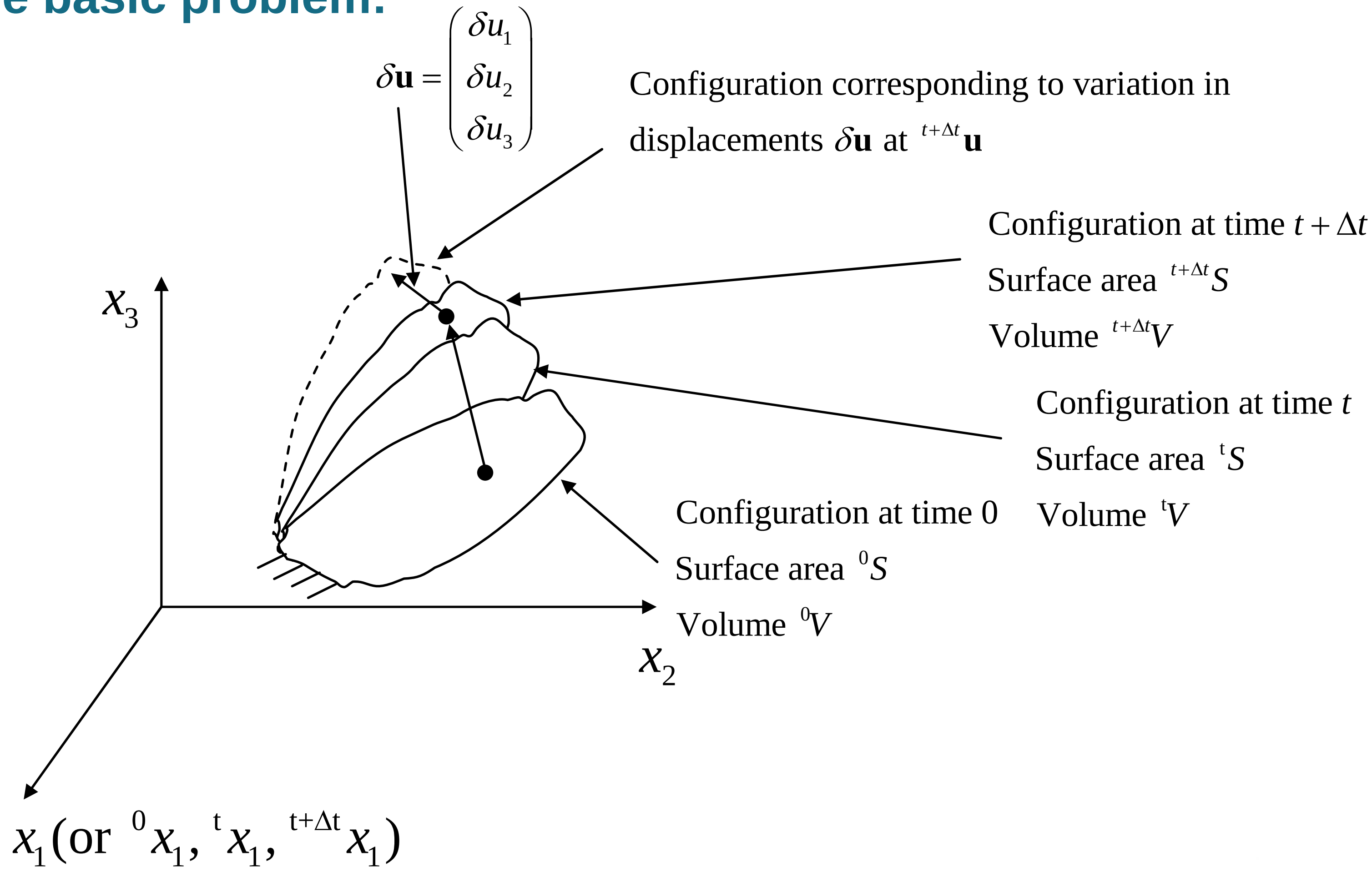
  In proceeding we adopt a Lagrangian formulation where we track the movement of all particles of the body (located in a Cartesian coordinate system)

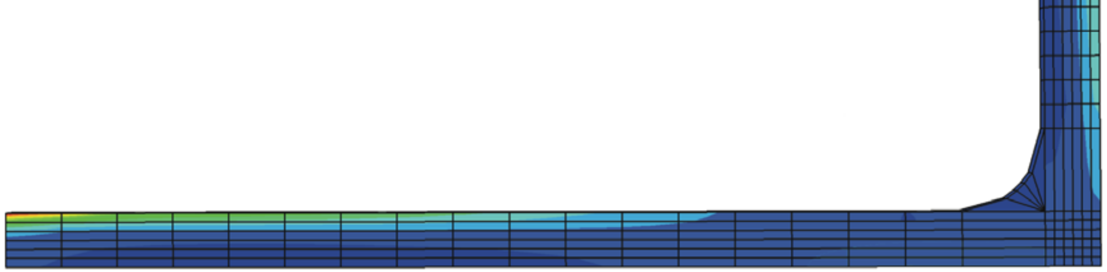
  Another approach would be an Eulerian formulation where the motion of material through a stationary control volume is considered



The continuum mechanics incremental equations

- **The basic problem:**

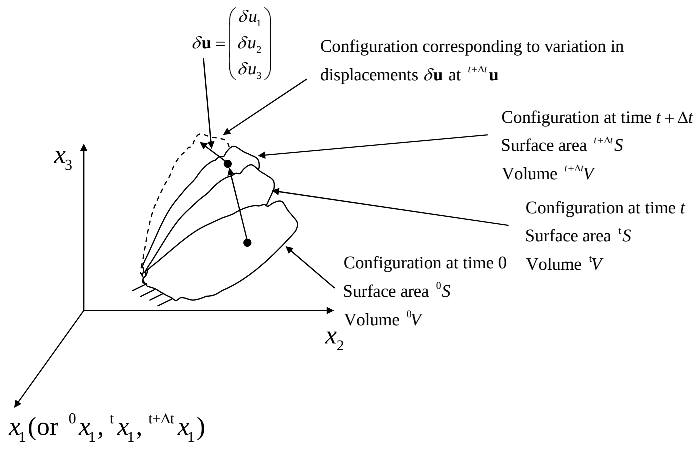




The continuum mechanics incremental equations

- ## The Lagrangian formulation

We express equilibrium of the body at time *t+Δt* using the principle of virtual displacements



$$\int_{{}^{t+\Delta t}V} {}^{t+\Delta t}\tau \, \delta_{t+\Delta t} e_{ij} \, d\,{}^{t+\Delta t}V = {}^{t+\Delta t}R$$

${}^{t+\Delta t}\tau$: Cartesian components of the Cauchy stress tensor

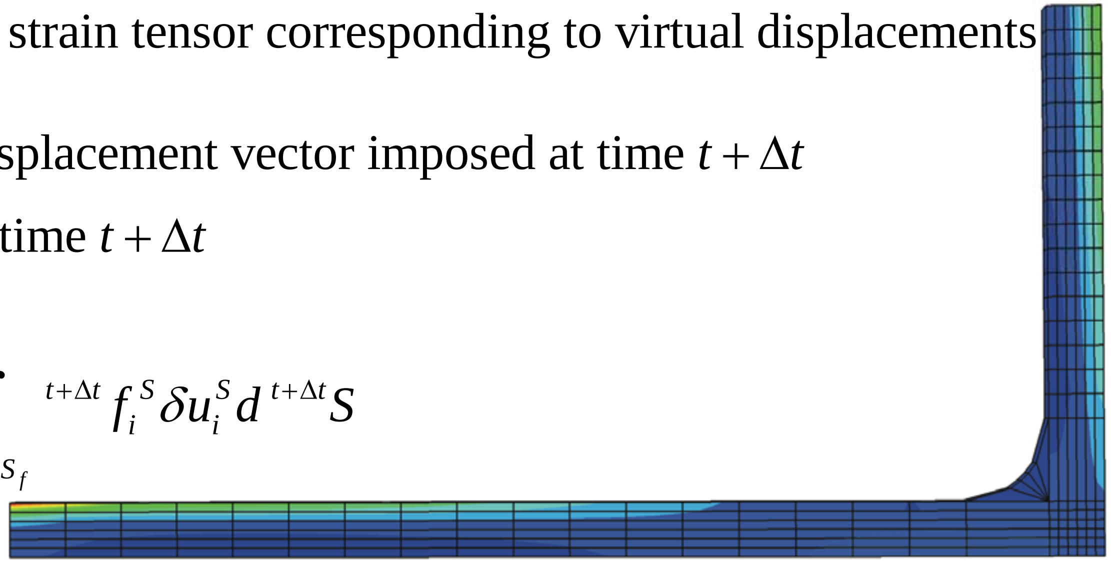
$$\delta_{t+\Delta t} e_{ij} = \frac{1}{2}\left(\frac{\partial \delta u_i}{\partial\,{}^{t+\Delta t}x_j} + \frac{\partial \delta u_j}{\partial\,{}^{t+\Delta t}x_i}\right) = \text{strain tensor corresponding to virtual displacements}$$

δu_i : Components of virtual displacement vector imposed at time $t+\Delta t$

${}^{t+\Delta t}x_i$: Cartesian coordinate at time $t+\Delta t$

${}^{t+\Delta t}V$: Volume at time $t+\Delta t$

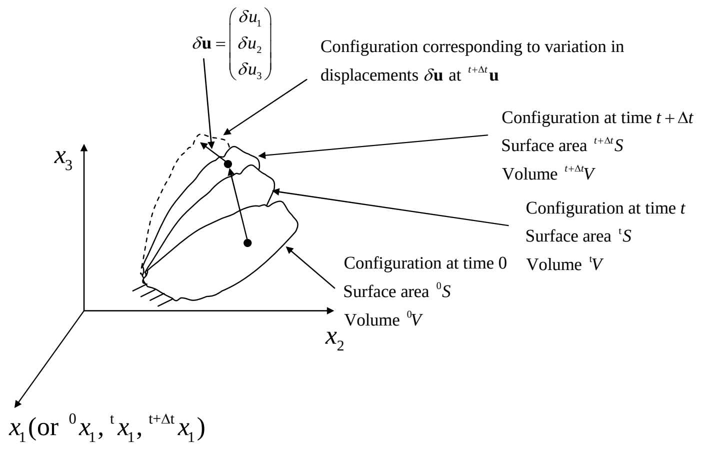
$$ {}^{t+\Delta t}R = \int_{{}^{t+\Delta t}V} {}^{t+\Delta t}f_i^B \, \delta u_i \, d\,{}^{t+\Delta t}V + \int_{{}^{t+\Delta t}S_f} {}^{t+\Delta t}f_i^S \, \delta u_i^S \, d\,{}^{t+\Delta t}S $$



The continuum mechanics incremental equations

- ## The Lagrangian formulation

We express equilibrium of the body at time *t+Δt* using the principle of virtual displacements



$$ {}^{t+\Delta t}R = \int_{{}^{t+\Delta t}V} {}^{t+\Delta t}f_i^B \, \delta u_i \, d\,{}^{t+\Delta t}V + \int_{{}^{t+\Delta t}S_f} {}^{t+\Delta t}f_i^S \, \delta u_i^S \, d\,{}^{t+\Delta t}S $$

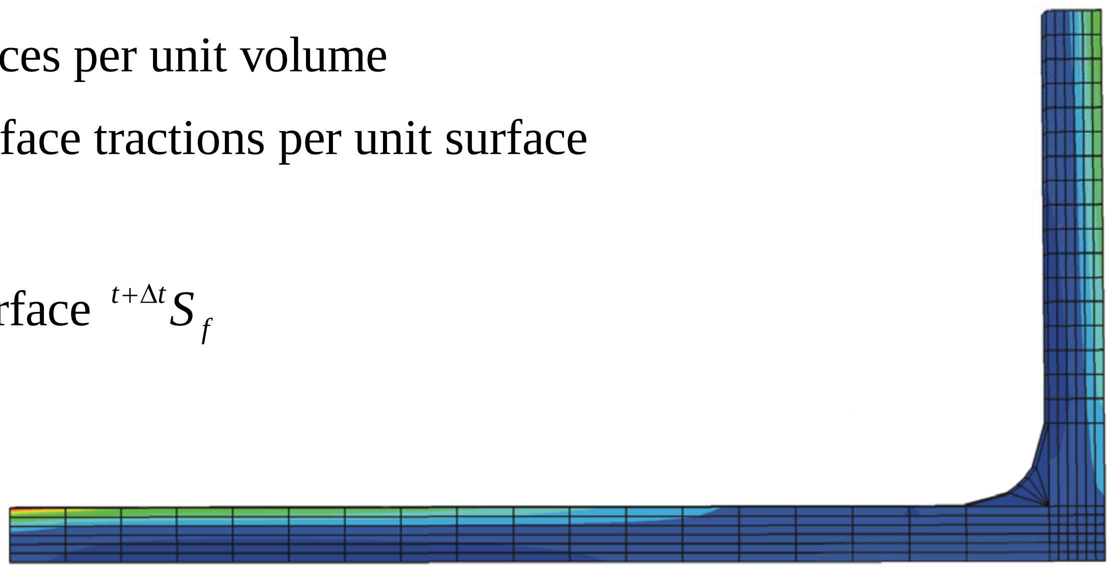
where

${}^{t+\Delta t}f_i^B$: externally applied forces per unit volume

${}^{t+\Delta t}f_i^S$: externally applied surface tractions per unit surface

${}^{t+\Delta t}S_f$: surface at time $t+\Delta t$

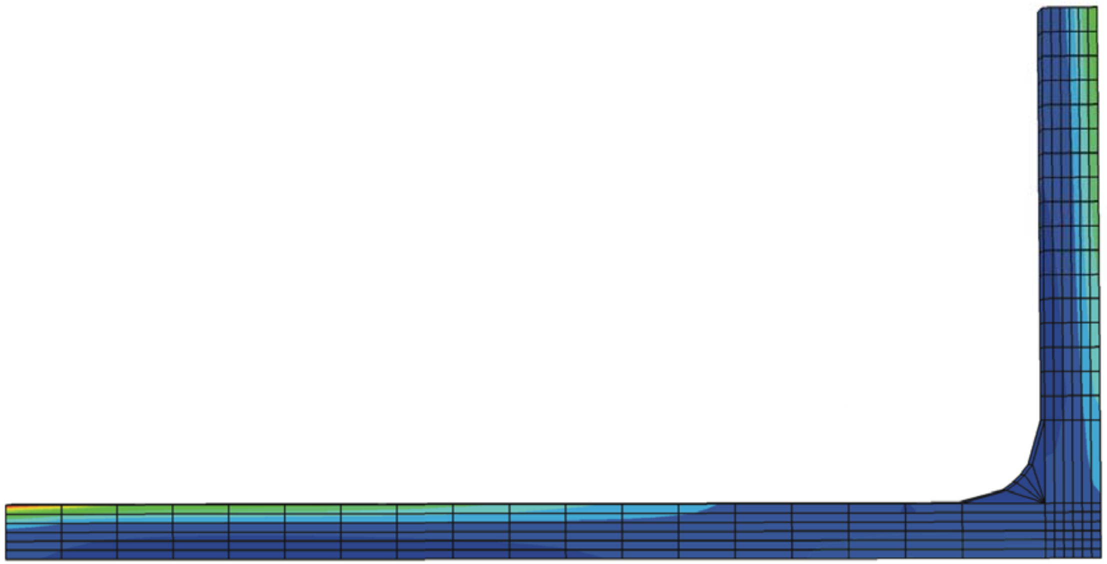
δu_i^S : δu_i evaluated at the surface ${}^{t+\Delta t}S_f$



The continuum mechanics incremental equations

- **The Lagrangian formulation**

  We recognize that our derivations from linear finite element theory are unchanged – but applied to the body in the configuration at time $t+\Delta t$



The continuum mechanics incremental equations

- **In the further we introduce an appropriate notation:**

Coordinates and displacements are related as:

$$ {}^{t}x_i = {}^{0}x_i + {}^{t}u_i $$

$$ {}^{t+\Delta t}x_i = {}^{0}x_i + {}^{t+\Delta t}u_i $$

Increments in displacements are related as:

$$ {}_{t}u_i = {}^{t+\Delta t}u_i - {}^{t}u_i $$

Reference configurations are indexed as e.g.:

${}^{t+\Delta t}_{0}f_i^{S}$ where the lower left index indicates the reference configuration

$$ {}^{t+\Delta t}\tau_{ij} = {}^{t+\Delta t}_{t+\Delta t}\tau_{ij} $$

Differentiation is indexed as:

$$ {}^{t+\Delta t}_{0}u_{i,j} = \frac{\partial\, {}^{t+\Delta t}u_i}{\partial\, {}^{0}x_j}, \qquad {}^{0}_{t+\Delta t}x_{m,n} = \frac{\partial\, {}^{0}x_m}{\partial\, {}^{t+\Delta t}x_n} $$