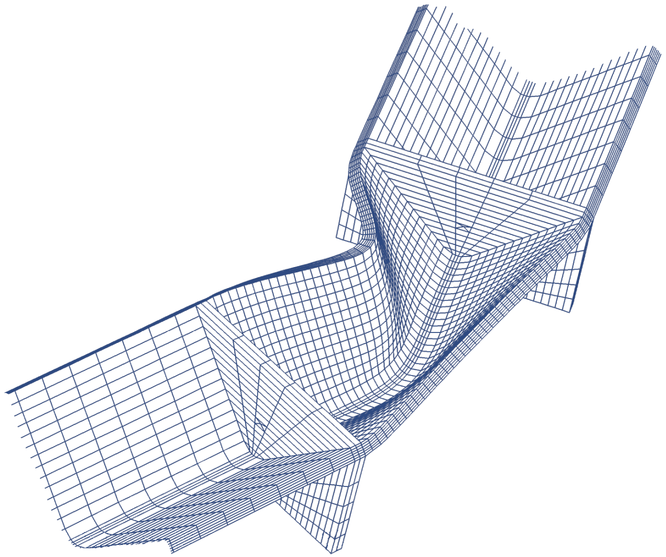
The Finite Element Method
for the Analysis of Non-Linear and Dynamic Systems



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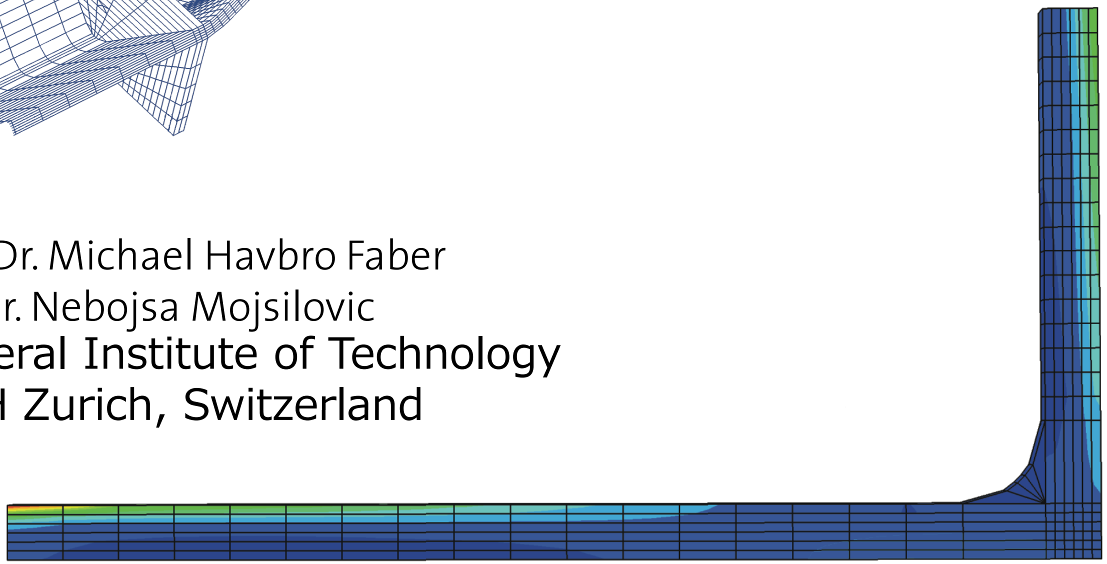


Contents of Today's Lecture

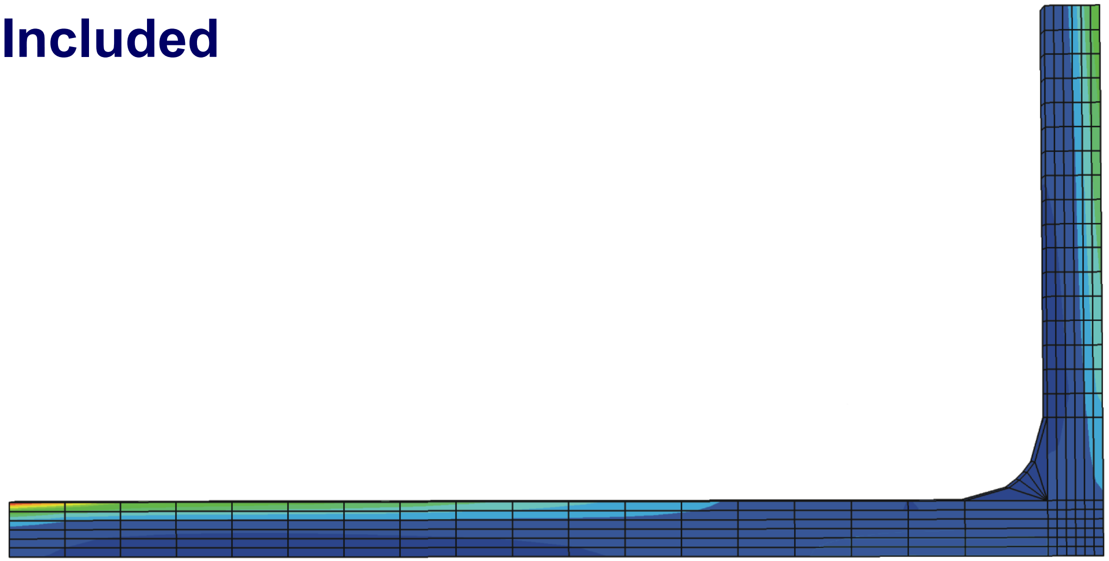
- **Solution of Equilibrium Equations in Dynamic Analysis**

  Mode Superposition

  Modal Generalized Displacements

  Analysis with Damping Neglected

  Analysis with Damping Included



Mode Superposition

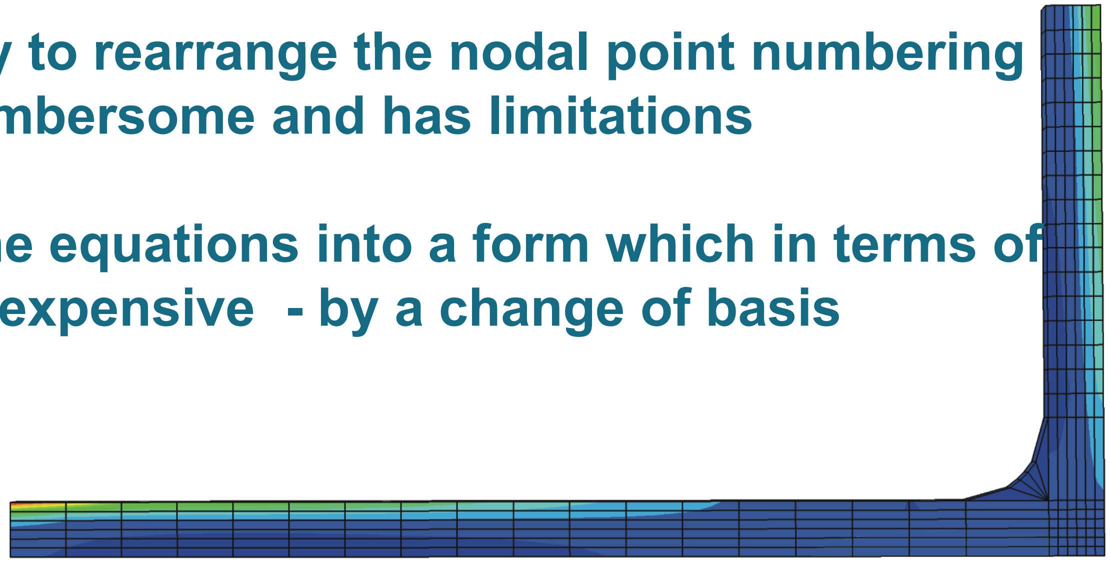
- **Modal Generalized Displacements**

  The direct integration methods necessitate that the finite element equations are evaluated for each time step

  The bandwidth of the matrixes M, C and K depend on the numbering of the finite element nodal points

  In principle we could try to rearrange the nodal point numbering but this approach is cumbersome and has limitations

  Instead we transform the equations into a form which in terms of numerical effort is less expensive - by a change of basis



Mode Superposition

- **Change of Basis to Modal Coordinates**

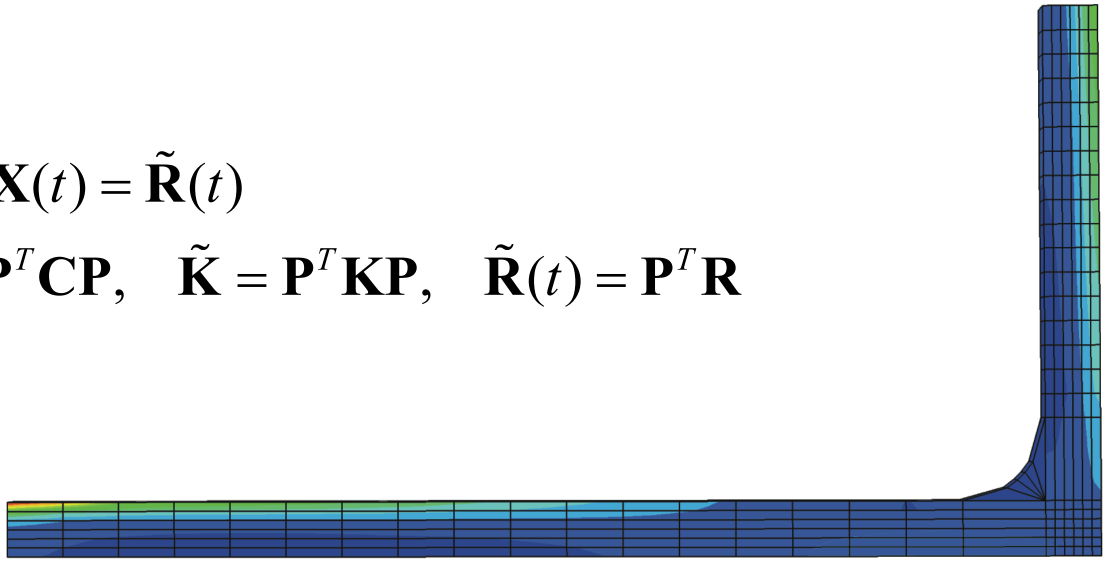
  The following transformation is introduced:

$$\mathbf{U}(t) = \mathbf{P}\mathbf{X}(t)$$

P: *n* **x** *n* **square matrix**
X(t): **time dependent vector of order n**

$$\tilde{\mathbf{M}}\ddot{\mathbf{X}}(t) + \tilde{\mathbf{C}}\dot{\mathbf{X}}(t) + \tilde{\mathbf{K}}\mathbf{X}(t) = \tilde{\mathbf{R}}(t)$$

$$\tilde{\mathbf{M}} = \mathbf{P}^T\mathbf{M}\mathbf{P}, \quad \tilde{\mathbf{C}} = \mathbf{P}^T\mathbf{C}\mathbf{P}, \quad \tilde{\mathbf{K}} = \mathbf{P}^T\mathbf{K}\mathbf{P}, \quad \tilde{\mathbf{R}}(t) = \mathbf{P}^T\mathbf{R}$$



Mode Superposition

$$\tilde{\mathbf{M}}\ddot{\mathbf{X}}(t)+\tilde{\mathbf{C}}\dot{\mathbf{X}}(t)+\tilde{\mathbf{K}}\mathbf{X}(t)=\tilde{\mathbf{R}}(t)$$

$$\tilde{\mathbf{M}}=\mathbf{P}^T\mathbf{M}\mathbf{P},\quad \tilde{\mathbf{C}}=\mathbf{P}^T\mathbf{C}\mathbf{P},\quad \tilde{\mathbf{K}}=\mathbf{P}^T\mathbf{K}\mathbf{P},\quad \tilde{\mathbf{R}}(t)=\mathbf{P}^T\mathbf{R}$$

- **Change of Basis to Modal Coordinates**

  The question is – how to choose P?

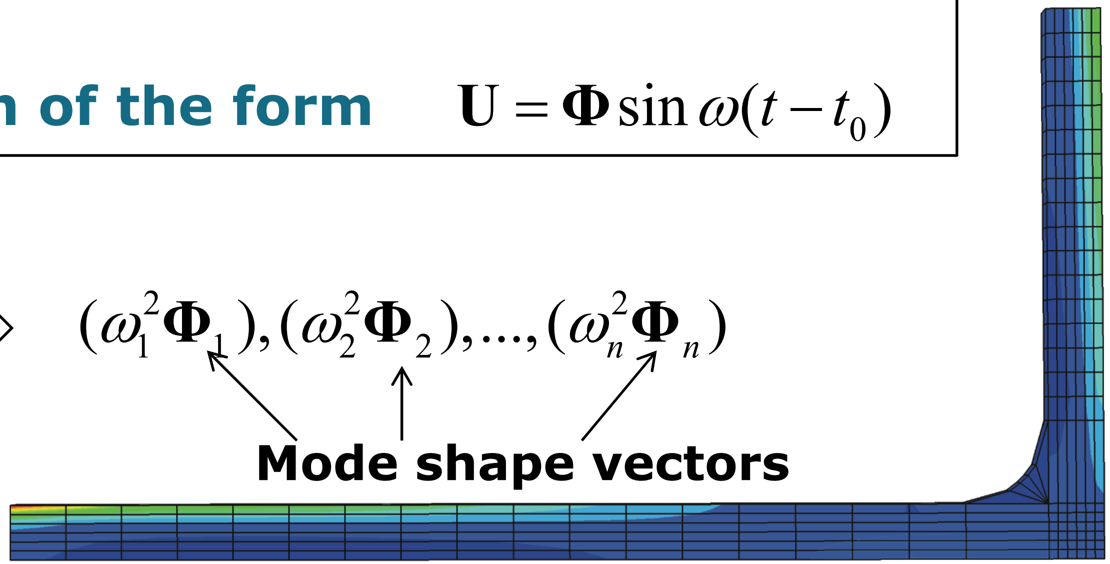
  A good choice is to take basis in the free vibration solution – neglecting damping, i.e.:

$$\mathbf{M}\ddot{\mathbf{U}}+\mathbf{K}\mathbf{U}=0$$

which has a solution of the form $\mathbf{U}=\boldsymbol{\Phi}\sin\omega(t-t_0)$

$$\mathbf{K}\boldsymbol{\Phi}_i=\omega_i^2\mathbf{M}\boldsymbol{\Phi}_i \quad\Rightarrow\quad (\omega_1^2\boldsymbol{\Phi}_1),(\omega_2^2\boldsymbol{\Phi}_2),\ldots,(\omega_n^2\boldsymbol{\Phi}_n)$$

Mode shape vectors



Mode Superposition

- **Change of Basis to Modal Coordinates**

Any of the solutions $(\omega_1^2 \mathbf{\Phi}_1), (\omega_2^2 \mathbf{\Phi}_2), \ldots, (\omega_n^2 \mathbf{\Phi}_n)$

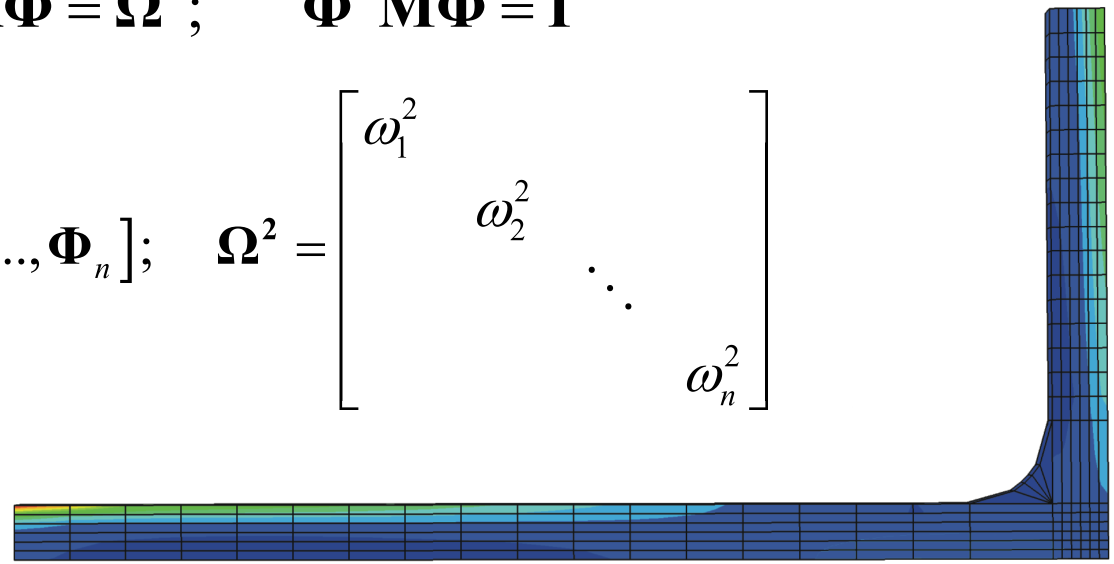
satisfy $\mathbf{M}\ddot{\mathbf{U}} + \mathbf{K}\mathbf{U} = 0$

The *n* solutions may be written as:

$$\mathbf{K}\mathbf{\Phi} = \mathbf{M}\mathbf{\Phi}\mathbf{\Omega}^2, \qquad \mathbf{\Phi}^T\mathbf{K}\mathbf{\Phi} = \mathbf{\Omega}^2; \qquad \mathbf{\Phi}^T\mathbf{M}\mathbf{\Phi} = \mathbf{I}$$

with:

$$\mathbf{\Phi} = [\mathbf{\Phi}_1, \mathbf{\Phi}_2, \ldots, \mathbf{\Phi}_n]; \quad \mathbf{\Omega}^2 = \begin{bmatrix} \omega_1^2 & & & \\ & \omega_2^2 & & \\ & & \ddots & \\ & & & \omega_n^2 \end{bmatrix}$$



Mode Superposition

- **Change of Basis to Modal Coordinates**

Now using $\mathbf{U}(t) = \mathbf{\Phi}\mathbf{X}(t)$

in

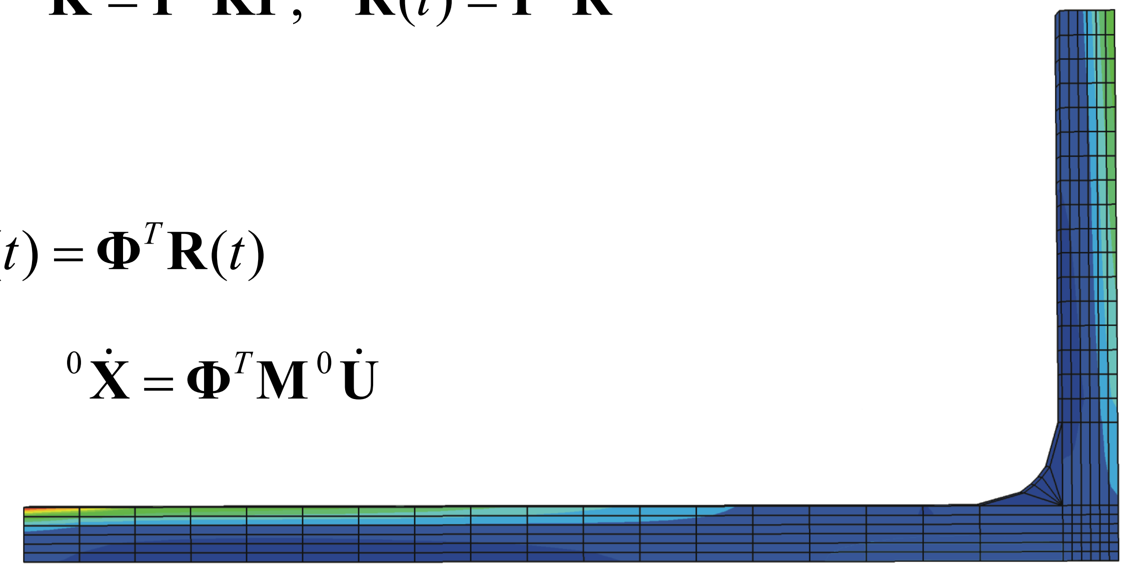
$$\tilde{\mathbf{M}}\ddot{\mathbf{X}}(t) + \tilde{\mathbf{C}}\dot{\mathbf{X}}(t) + \tilde{\mathbf{K}}\mathbf{X}(t) = \tilde{\mathbf{R}}(t)$$

$$\tilde{\mathbf{M}} = \mathbf{P}^T\mathbf{M}\mathbf{P}, \quad \tilde{\mathbf{C}} = \mathbf{P}^T\mathbf{C}\mathbf{P}, \quad \tilde{\mathbf{K}} = \mathbf{P}^T\mathbf{K}\mathbf{P}, \quad \tilde{\mathbf{R}}(t) = \mathbf{P}^T\mathbf{R}$$

we get

$$\ddot{\mathbf{X}}(t) + \mathbf{\Phi}^T\mathbf{C}\mathbf{\Phi}\dot{\mathbf{X}}(t) + \mathbf{\Omega}^2\mathbf{X}(t) = \mathbf{\Phi}^T\mathbf{R}(t)$$

with ${}^0\mathbf{X} = \mathbf{\Phi}^T\mathbf{M}\,{}^0\mathbf{U}; \qquad {}^0\dot{\mathbf{X}} = \mathbf{\Phi}^T\mathbf{M}\,{}^0\dot{\mathbf{U}}$



Mode Superposition

- **Analysis with damping neglected**

Here we start with: $\ddot{\mathbf{X}}(t) + \mathbf{\Omega}^2 \mathbf{X}(t) = \mathbf{\Phi}^T \mathbf{R}(t)$

i.e.:

$$\left.\begin{aligned} &\ddot{x}_i(t) + \omega_i^2 x_i(t) = r_i(t) \\ &r_i(t) = \mathbf{\Phi}_i^T \mathbf{R}(t) \end{aligned}\right\}$$

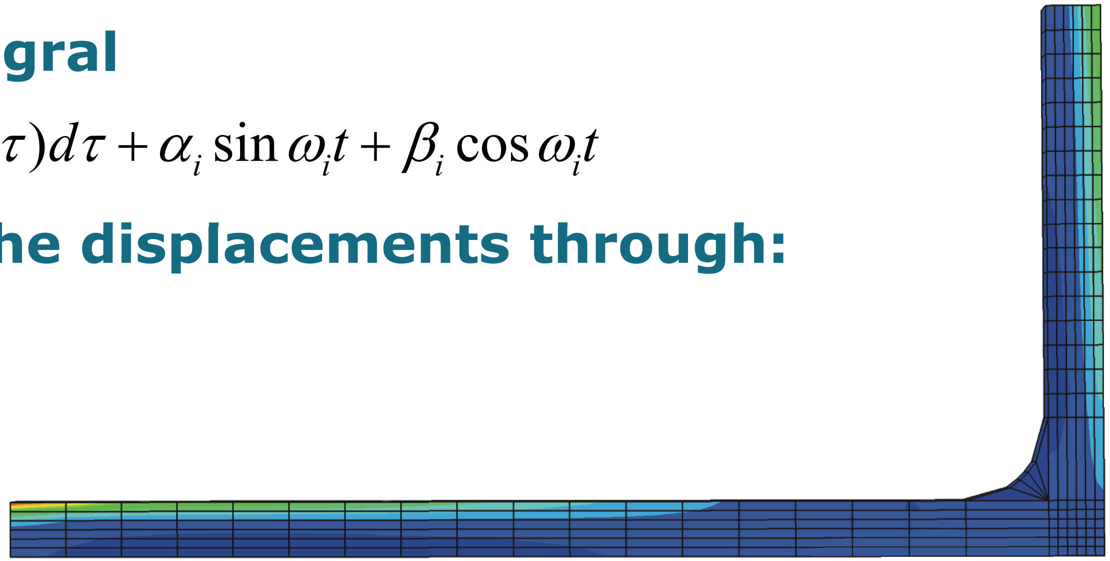
Can be solved using the direct integration schemes

or the Duhamel integral

$$x_i(t) = \frac{1}{\omega_i} \int_0^t r_i(\tau) \sin \omega_i\,(t-\tau)\,d\tau + \alpha_i \sin \omega_i t + \beta_i \cos \omega_i t$$

and we convert to the displacements through:

$$\mathbf{U}(t) = \sum_{i=1}^{n} \mathbf{\Phi}_i x_i(t)$$



Mode Superposition

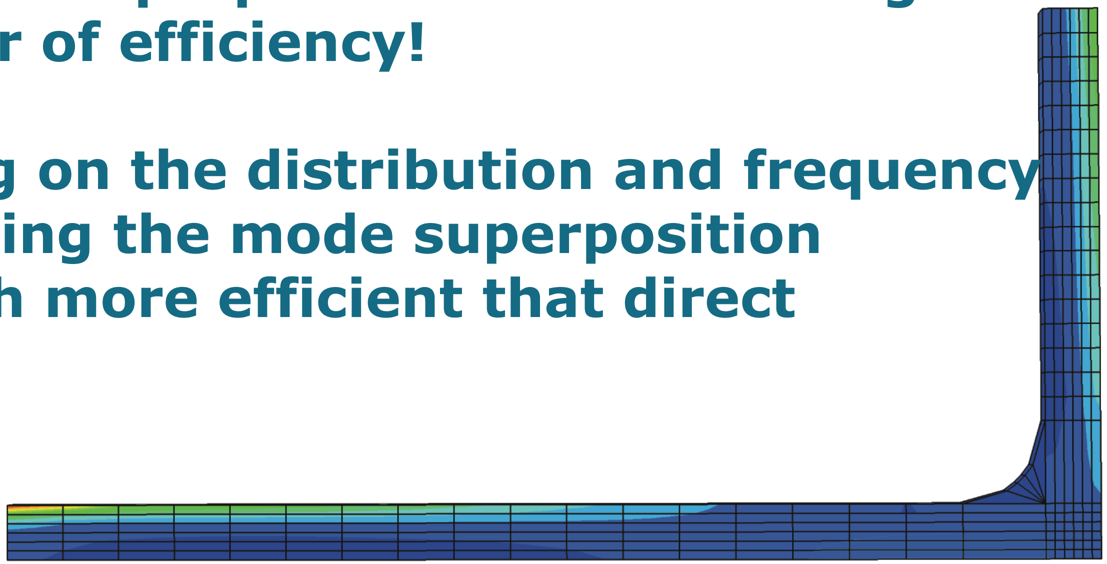
- **Analysis with damping neglected**

  Comparing mode superposition with direct integration we have so far only changed the basis before integrating

  The solutions must thus be the same!

  Whether to use mode superposition or direct integration is thus only a matter of efficiency!

  However, depending on the distribution and frequency contents of the loading the mode superposition method can be much more efficient that direct integration



Mode Superposition

- **Analysis with damping neglected**

Considering again

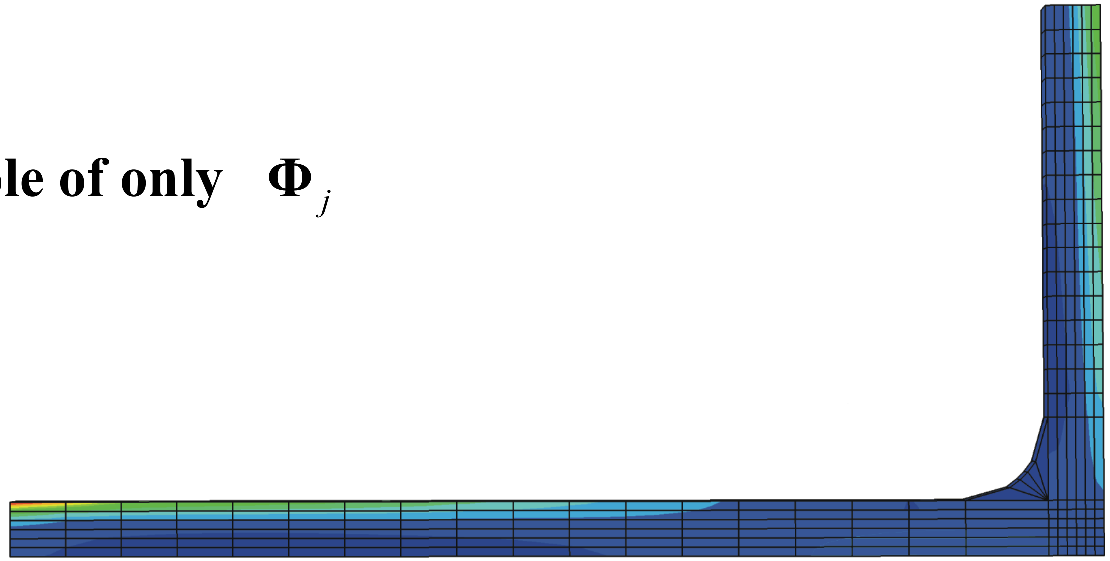
$$\ddot{x}_i(t) + \omega_i^2 x_i(t) = r_i(t), \qquad r_i(t) = \boldsymbol{\Phi}_i^T \mathbf{R}(t)$$

and setting $r_i(t) = 0, \qquad i = 1, 2, \ldots, n$

and either

${}^0\mathbf{U}$ **or** ${}^0\dot{\mathbf{U}}$ **are a multiple of only** $\boldsymbol{\Phi}_j$

then only $x_j(t) \neq 0$



Mode Superposition

- **Analysis with damping neglected**

if instead we set $\,^{0}\mathbf{U} = \,^{0}\dot{\mathbf{U}} = \mathbf{0}$ **and** $\mathbf{R}(t) = \mathbf{M}\boldsymbol{\Phi}_j f(t)$

then only $x_j(t) \neq 0$

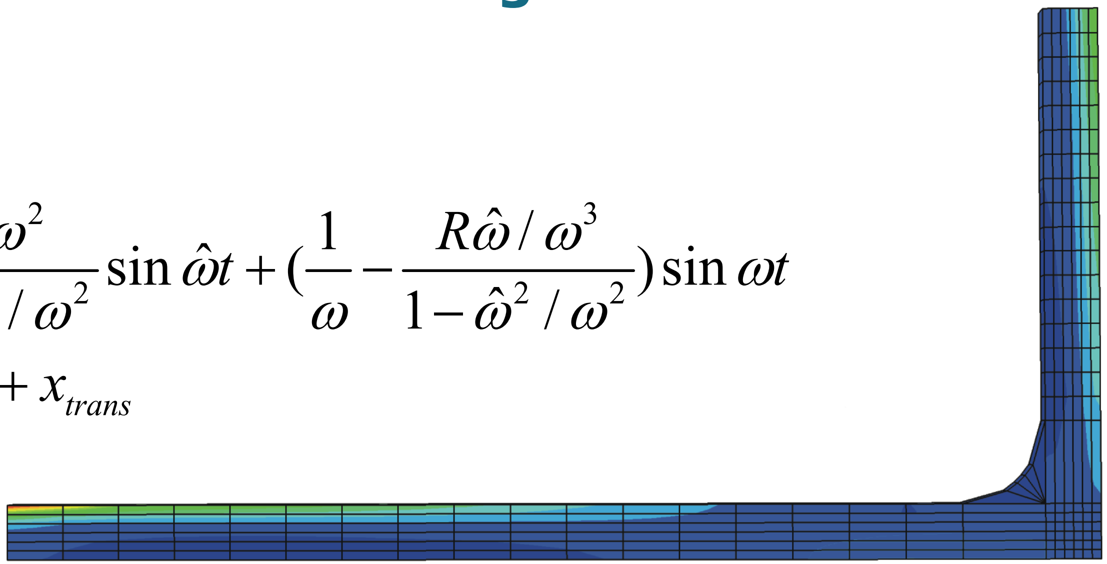
Example 9.8 shows that for a one degree of freedom system

$$\ddot{x}(t) + \omega^2 x\ (t) = R \sin \hat{\omega} t$$

there is:

$$x(t) = \frac{R/\omega^2}{1-\hat{\omega}^2/\omega^2} \sin \hat{\omega} t + \left(\frac{1}{\omega} - \frac{R\hat{\omega}/\omega^3}{1-\hat{\omega}^2/\omega^2}\right) \sin \omega t$$

$$= D x_{stat} + x_{trans}$$



Mode Superposition

$$\ddot{x}(t) + \omega^2 x\ (t) = R \sin \hat{\omega} t$$

$$x(t) = \frac{R/\omega^2}{1-\hat{\omega}^2/\omega^2} \sin \hat{\omega} t + \left(\frac{1}{\omega} - \frac{R\hat{\omega}/\omega^3}{1-\hat{\omega}^2/\omega^2}\right) \sin \omega t$$

$$= D x_{stat} + x_{trans}$$

- **Analysis with damping neglected**

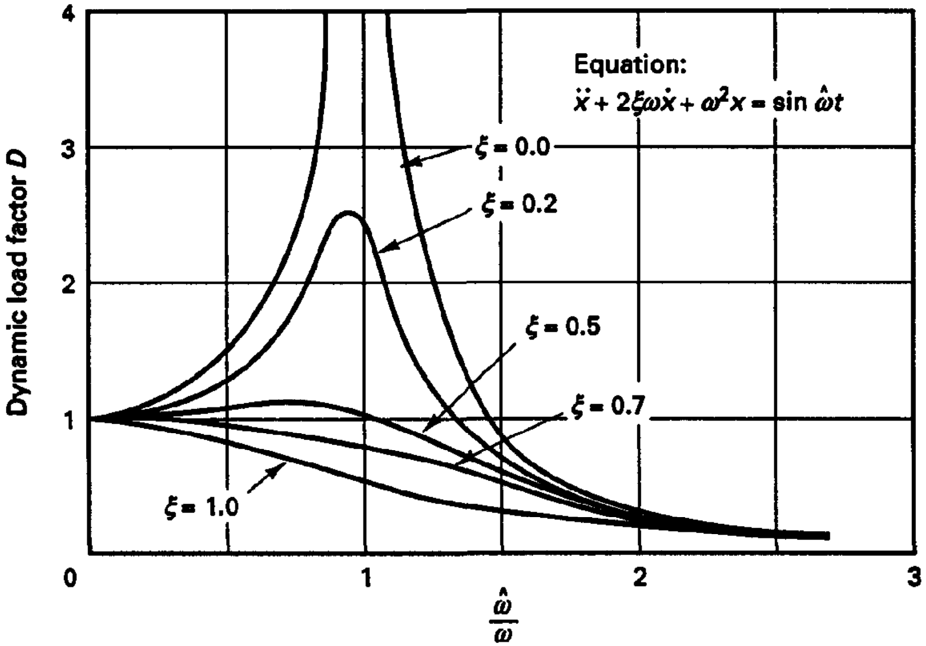


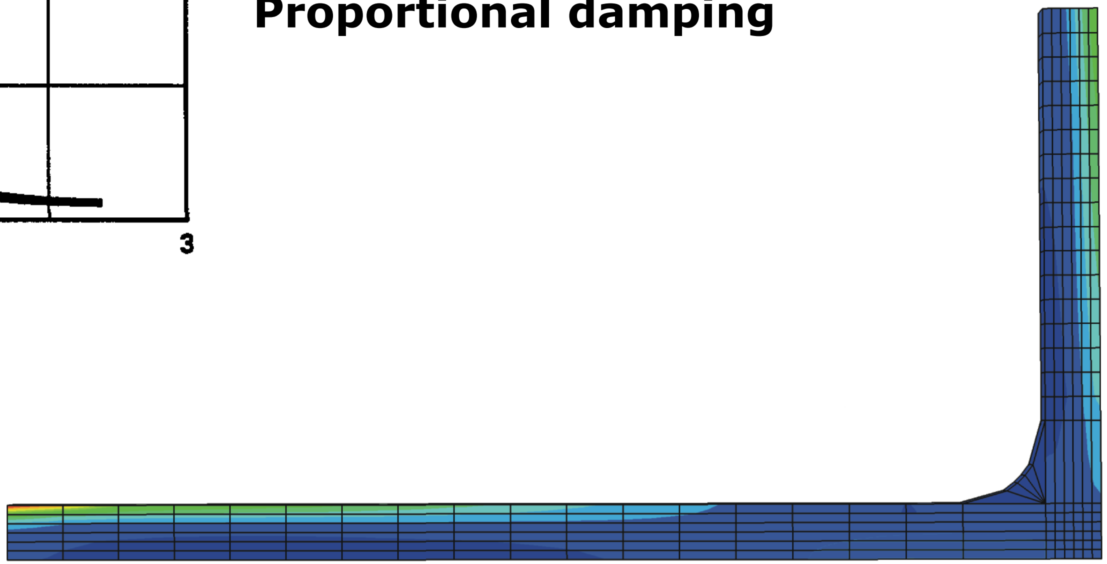
Figure 9.3 The dynamic load factor

$$\Phi_i^T \mathbf{C} \Phi_i = 2\omega_i \zeta_i \delta_{ij}$$

ζ_i = **modal damping parameters**

δ_{ij} = **Kronecker delta** $\delta_{ij} = 1$, **for** $i = j$ **else** $\delta_{ij} = 1$

Proportional damping



Mode Superposition

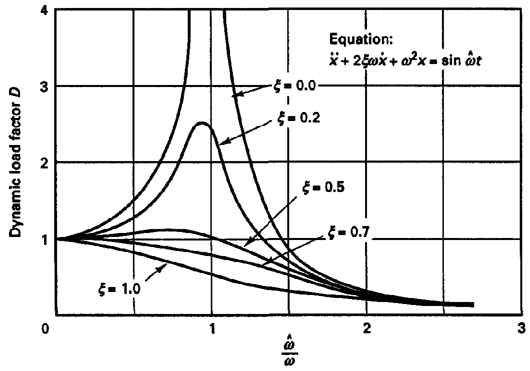


Figure 9.3 The dynamic load factor

- **Analysis with damping neglected**

Considering again

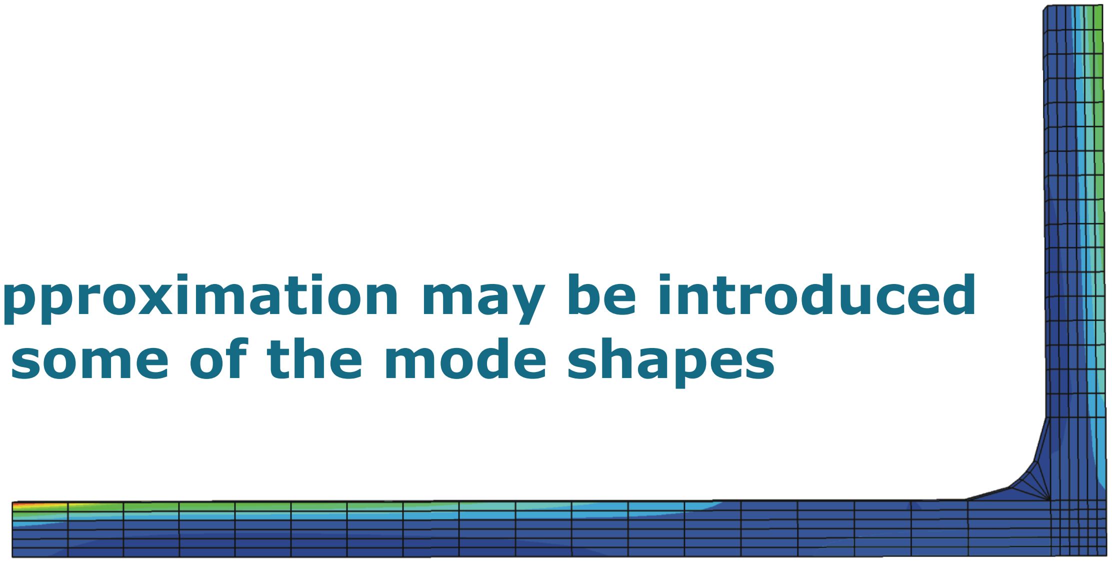
$$\ddot{x}_i(t) + \omega_i^2 x_i(t) = r_i(t)$$

$$r_i(t) = \mathbf{\Phi}_i^T \mathbf{R}(t)$$

$$x_i(t) = \frac{1}{\omega_i}\int_0^t r_i(\tau)\sin\omega_i\,(t-\tau)\,d\tau + \alpha_i \sin\omega_i t + \beta_i \cos\omega_i t$$

$$\mathbf{U}(t) = \sum_{i=1}^{n} \mathbf{\Phi}_i x_i(t)$$

we realize that an approximation may be introduced by only considering some of the mode shapes



Mode Superposition

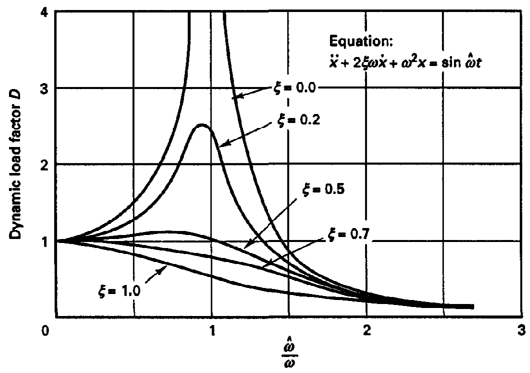


Figure 9.3 The dynamic load factor

- **Analysis with damping neglected**

Considering again

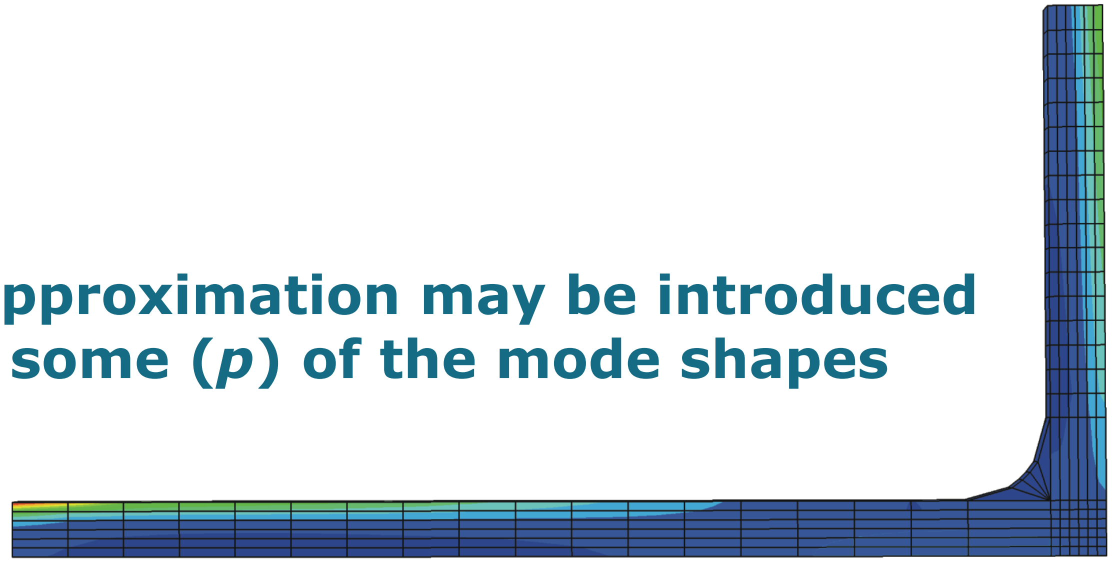
$$\ddot{x}_i(t) + \omega_i^2 x_i(t) = r_i(t)$$

$$r_i(t) = \mathbf{\Phi}_i^T \mathbf{R}(t)$$

$$x_i(t) = \frac{1}{\omega_i}\int_0^t r_i(\tau)\sin\omega_i\,(t-\tau)\,d\tau + \alpha_i \sin\omega_i t + \beta_i \cos\omega_i t$$

$$\mathbf{U}^p(t) = \sum_{i=1}^{p} \mathbf{\Phi}_i x_i(t)$$

we realize that an approximation may be introduced by only considering some (*p*) of the mode shapes



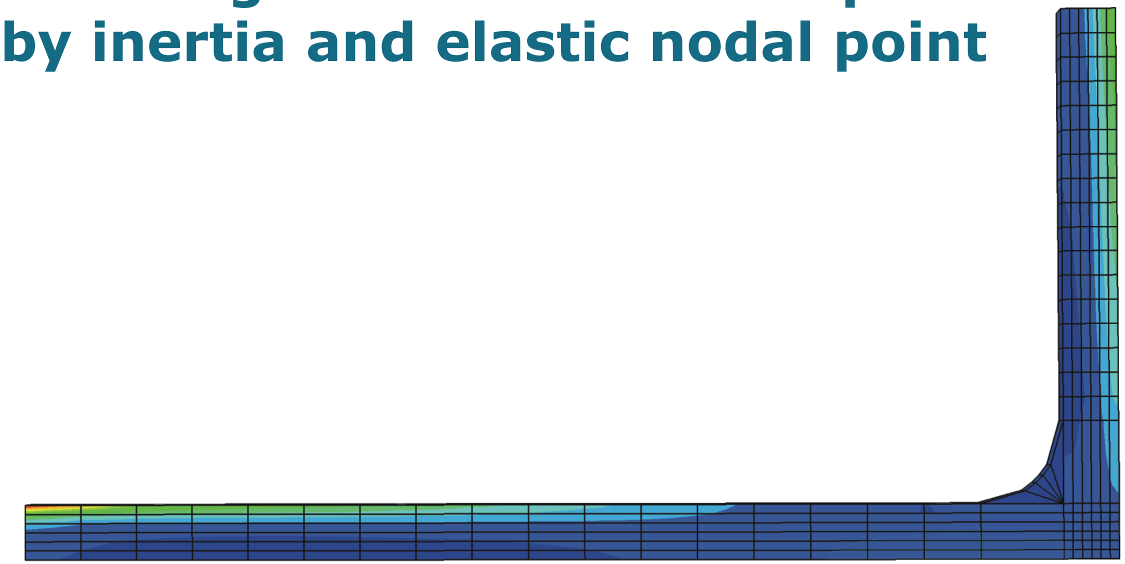
Mode Superposition

- **Analysis with damping neglected**

The error made may then be estimated as:

$$\varepsilon^{p}(t)=\frac{\left\|\mathbf{R}(t)-\left[\mathbf{M}\ddot{\mathbf{U}}^{p}(t)+\mathbf{K}\mathbf{U}^{p}(t)\right]\right\|_{2}}{\left\|\mathbf{R}(t)\right\|_{2}}$$

This is a measure of the degree to which nodal point loads are balanced by inertia and elastic nodal point forces.



Mode Superposition

- **Analysis with damping neglected**

The unbalance in nodal forces is:

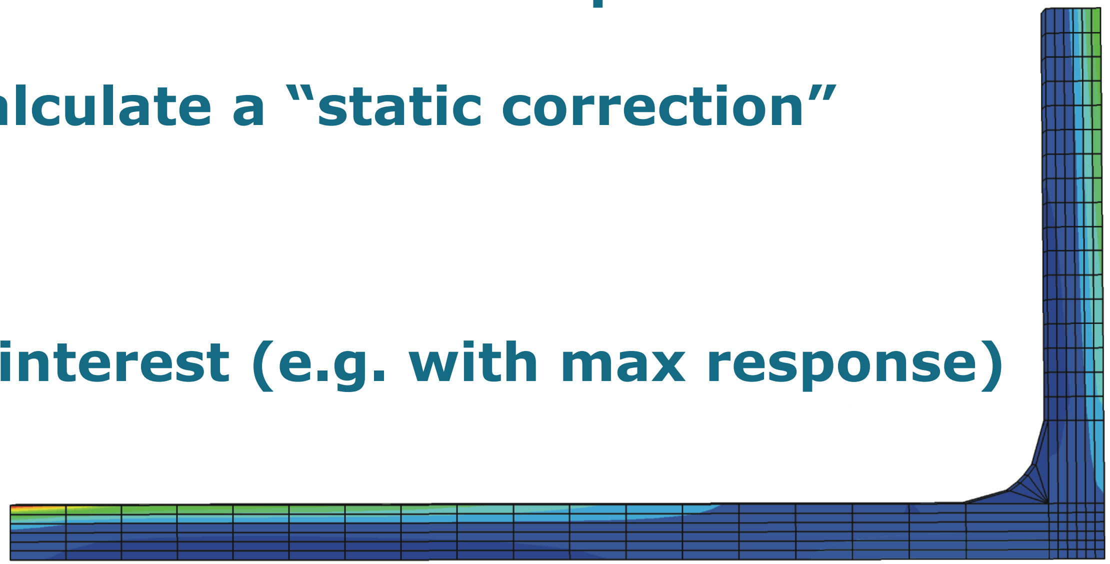
$$\Delta \mathbf{R} = \mathbf{R} - \sum_{i=1}^{p} r_i (\mathbf{M} \boldsymbol{\Phi}_i)$$

If the problem is modeled appropriately the unbalance should at most amount to the static response

Therefore we can calculate a "static correction"

$$\mathbf{K} \Delta \mathbf{U}(t) = \Delta \mathbf{R}(t)$$

for times of special interest (e.g. with max response)



Mode Superposition

- **Analysis with damping included**

In case of proportional damping e.g.:

$$\boldsymbol{\Phi}_i^T \mathbf{C} \boldsymbol{\Phi}_i = 2\omega_i \zeta_i \delta_{ij}$$

$\zeta_i =$ **modal damping parameters**

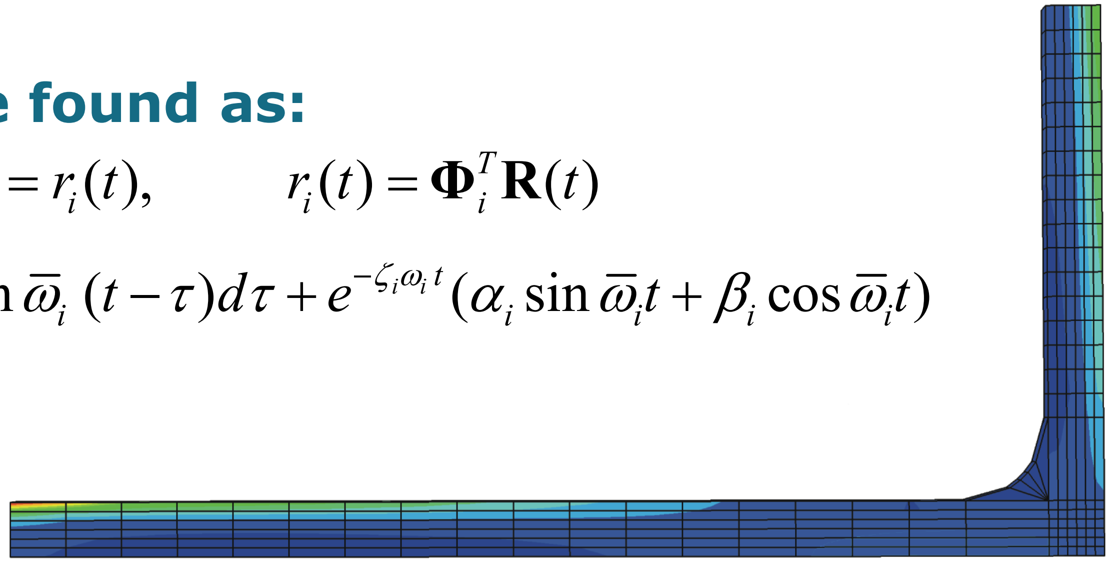
$\delta_{ij} =$ **Kronecker delta** $\delta_{ij} = 1$, **for** $i = j$ **else** $\delta_{ij} = 1$

The solution may be found as:

$$\ddot{x}_i(t) + 2\omega_i^2 \zeta_i \dot{x}_i(t) + \omega_i^2 x_i(t) = r_i(t), \qquad r_i(t) = \boldsymbol{\Phi}_i^T \mathbf{R}(t)$$

$$x_i(t) = \frac{1}{\omega_i} \int_0^t r_i(\tau) e^{-\zeta_i \omega_i (t-\tau)} \sin \bar{\omega}_i (t-\tau) d\tau + e^{-\zeta_i \omega_i t} (\alpha_i \sin \bar{\omega}_i t + \beta_i \cos \bar{\omega}_i t)$$

$$\bar{\omega}_i = \omega_i \sqrt{1 - \zeta_i^2}$$



Mode Superposition

- **Analysis with damping included**

  Assume that we know the damping ratios for a number of modes i.e.:

  $$\zeta_i, i = 1, 2, ..r$$

  If *r* = 2 Rayleigh damping can be used in the form:

  $$\mathbf{C} = \alpha \mathbf{M} + \beta \mathbf{K}$$

  Examples 9.9-911 illustrate how to use this approach.