Solution of Equilibrium Equation in Dynamic Analysis

Mode Superposition

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Contents

Dynamic Equilibrium Equation

$$M\ddot{U} + C\dot{U} + KU = R$$

Solving methods:

- Central difference
- Houbolt
- Wilson θ
- Newmark

Effectivity of this methods

Implicite methods

(Houbolt, Wilson, Newmark)

1.) Initial calculations (LDL^T) -> Number of operations: $$o = \frac{1}{2} \cdot n \cdot m_k^2$$

2.) For each time step (Mult.) -> Number of operations $$o = 2 \cdot n \cdot m_k$$

Where: n : matrix size, m_k: half bandwidth

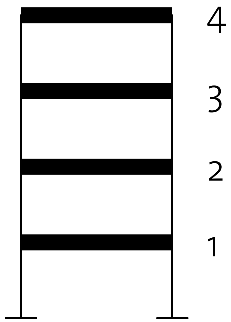
Number of operation is growing with bandwidth !

Reduction of bandwidth

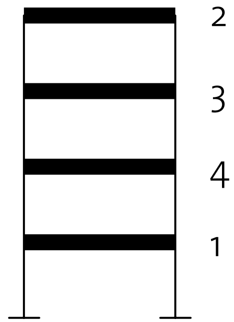
1.) Optimize mesh topology

-> Limited effect

$$K = k \cdot \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$



$$K = k \cdot \begin{bmatrix} 2 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$



2.) Transformation

Modal generalized displacements

Transformation with unknown P, such that m_k smaller

$$U(t) = P \cdot X(t)$$

Dynamic Equation in modal displacements

$$M\ddot{U} + C\dot{U} + KU = R$$

$$MP\ddot{X} + CP\dot{X} + KPX = R$$

$$\underbrace{P^T MP}_{\tilde{M}}\ddot{X} + \underbrace{P^T CP}_{\tilde{C}}\dot{X} + \underbrace{P^T KP}_{\tilde{K}}X = \underbrace{P^T R}_{\tilde{R}}$$

In theory many different transformations possible –

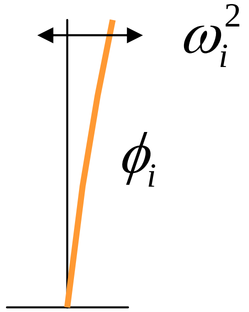
in practice only one transformation matrix established !

Transformation matrix

Free-vibration equilibrium solutions with damping neglected:

$$M\ddot{U} + KU = 0$$

$$U = \phi \cdot \sin\left(\omega \cdot \left(t - t_0\right)\right)$$ **Postulated solution**



By inserting U we obtain a eigenproblem:

$$K\phi - M\omega^2\phi = 0$$ **with n solutions for eigenvector** ϕ_i **eigenvalue** ω_i^2

Normalization:

$$\phi_i^T M \phi_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

$$0 \leq \omega_1^2 \leq \omega_2^2 \leq \ldots \leq \omega_n^2$$

Transformation matrix

Definitions

$$\Phi = [\phi_1, \phi_2, ..., \phi_n]$$

$$\Omega^2 = \begin{bmatrix} \omega_1^2 & & & \\ & \omega_1^2 & & \\ & & \dots & \\ & & & \omega_n^2 \end{bmatrix}$$

Eigenproblem for n equations

$$K\Phi - M\Phi\Omega^2 = 0$$

Since the eigenvectors are M - orthonormal

$$\Phi^T M\Phi = I \qquad \Phi^T K\Phi = \Omega^2$$

M and K diagonalized with TRANSFORMATION MATRIX Φ

$$\Phi = [\phi_1, \phi_2, ..., \phi_n] \qquad U(t) = \Phi \cdot X(t)$$

Transformed equation, Damping neglected

Matrix equation

$$M\ddot{U} + KU = 0$$

$$\underbrace{\Phi^T M \Phi}_{I} \ddot{X} + \underbrace{\Phi^T K \Phi}_{\Omega^2} X = \Phi^T R$$

$$\ddot{X} + \Omega^2 X = \Phi^T R$$

Single decoupled equation i

$$\ddot{x}_i(t) + \omega_i^2 \cdot x(t) = r_i(t) \qquad i = 1, \quad 2, \ldots., n$$

$$r_i(t) = \phi_i^T \cdot R(t)$$

Transforming the initial conditions

$$x_i(t=0) = \phi_i^T M\, {}^0U$$

Solution of decoupled equation i

Duhamel integral

$$x_i(t) = \frac{1}{\omega_i}\int_0^t r_i(\tau)\cdot\sin(\omega_i\cdot(t-\tau))d\tau + \alpha_i\cdot\sin(\omega_i\cdot t) + \beta_i\cdot\cos(\omega_i\cdot t)$$

Transforming to real displacement base

$$U(t) = \sum_{i=1}^{n}\phi_i\cdot x_i(t)$$

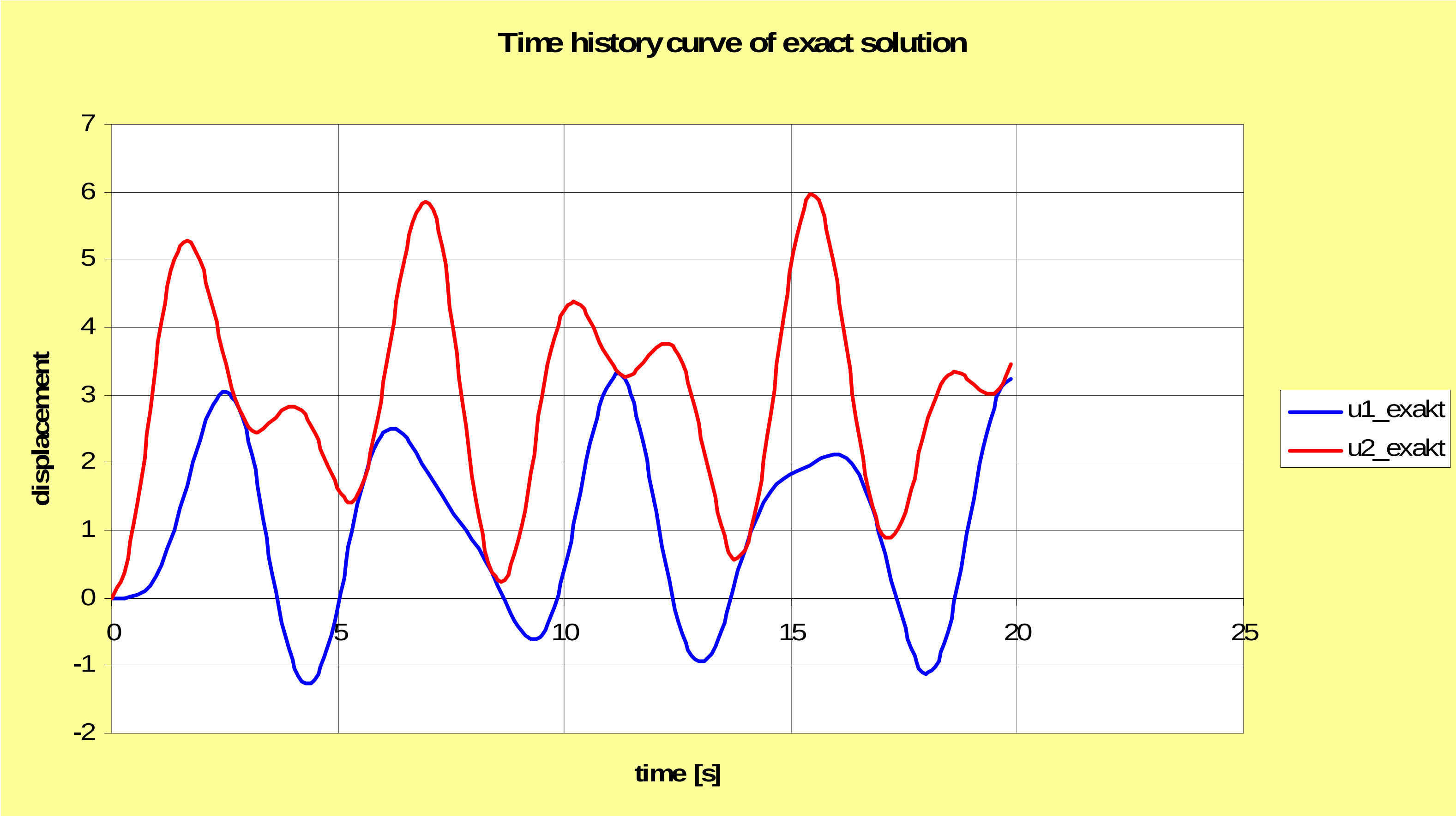
Example 9.7; p. 789

EXAMPLE 9.7: Use mode superposition to calculate the displacement response of the system considered in Examples 9.1 to 9.4 and 9.6.

(1) Calculate the exact response by integrating each of the two decoupled equilibrium equations exactly.

(2) Use the Newmark method with time step $\Delta t = 0.28$ for the time integration.

Exact solution with mode superposition



Newmark method

Two possibilities, leading to the same result

1.) Integrate $M\ddot{U} + KU = R$ straight forward -> SLOW

2.) Transform $M\ddot{U} + KU = R$ into modal displacements $\ddot{X} + \Omega^2 X = \Phi^T R$,
integrate the decoupled equations , transforme back -> FAST

2 Example

Newmark in MATLAB

$$\Delta t = 0.28$$

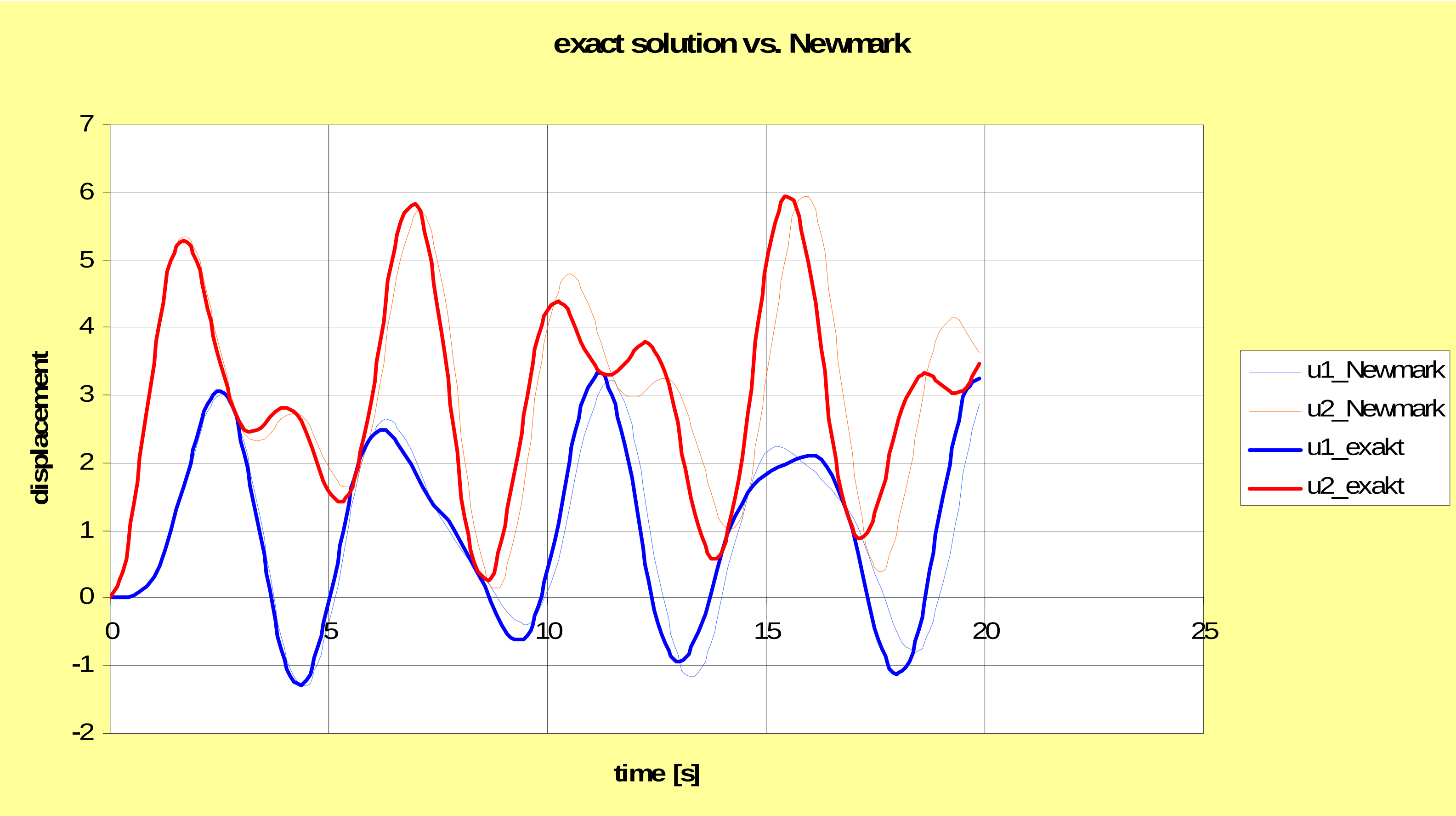
$$\delta = 0.5$$

$$\alpha = 0.25$$

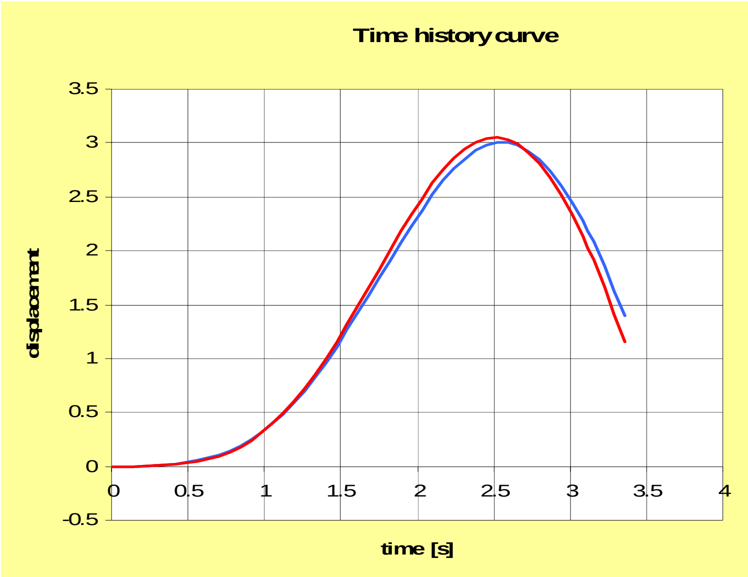
Originally proposed as an unconditionally stable scheme by Newmark

```
%INITIAL CALCULATIONS
%system
M = [2, 0; 0, 1];
K = [6, -2; -2,4];
C = [0, 0; 0, 0];
R = [0; 10];

%Initial conditions
U = [0; 0];
U1 = [0; 0];
U2 = inv(M)*(R - K*U - C*U1);

%parameters
n = 12;
dt = 0.28;
delta = 0.5;
alpha = 0.25;

%Integration constants
a0 = 1/(alpha*dt^2);
a1 = delta/(alpha*dt);
a2 = 1/(alpha*dt);
a3 = 1/(2*alpha)-1;
a4 = delta/alpha - 1;
a5 = dt/2*(delta/alpha-2);
a6 = dt*(1-delta);
a7 = delta*dt;

%stiffness matrix
Ks = K + a0*M + a1*C;

%Triangularization
%not worthy here
Ut = U;

%FOR EACH TIME STEP
for i = 1:n
    %effective loads at time t + dt
    Rs = R + M*(a0*U + a2*U1 + a3*U2) + C*(a1*U + a4*U1 + a5*U2);
    %solving displacements
    Ualt = U;
    U1alt = U1;
    U2alt = U2;
    U = inv(Ks)*Rs;
    %Backcalculating accs and velos
    U2 = a0*(U-Ualt) - a2*U1alt - a3*U2alt;
    U1 = U1alt + a6*U2alt + a7*U2;
    Ut = [Ut, U];
end
Ut
```

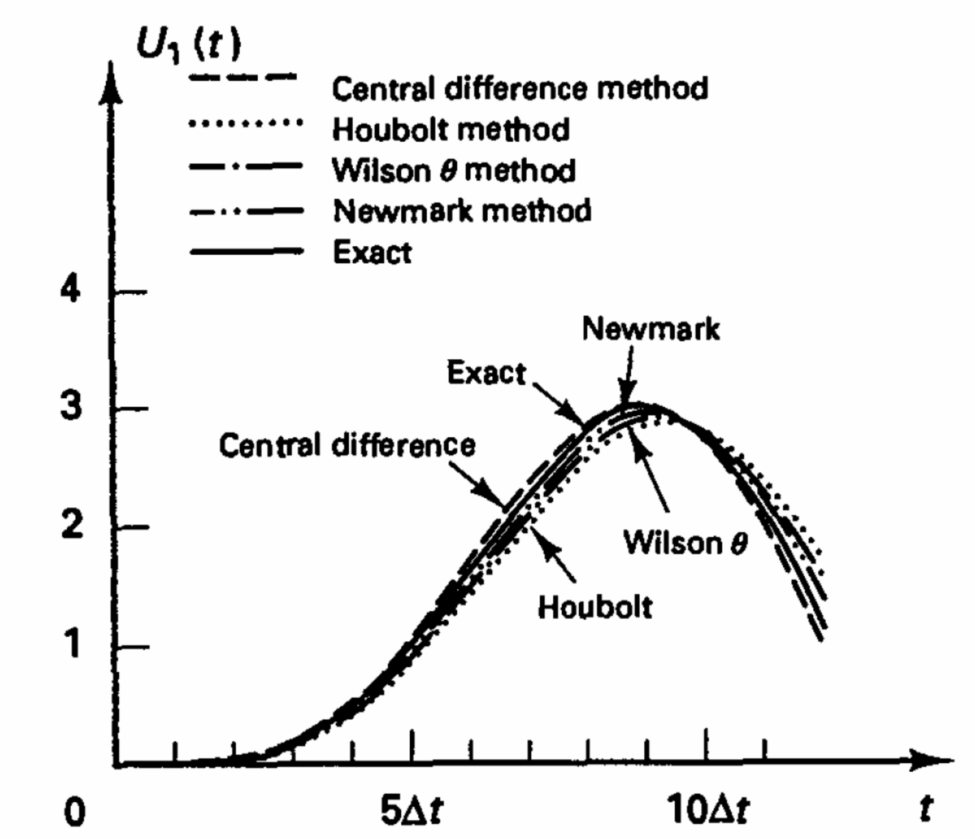
Newmark method



Benchmark



Time history curve
3.5
3
2.5
2
1.5
1
0.5
0
-0.5
displacement
0
0.5
1
1.5
2
2.5
3
3.5
4
time [s]



U1 (t)
Central difference method
Houbolt method
Wilson θ method
Newmark method
Exact
4
3
2
1
0
Newmark
Exact
Central difference
Wilson θ
Houbolt
5Δt
10Δt
t

Number of modes in calculation

For n lumped masses in a system, n modes were found - but for a good approximation often only a few are needed!

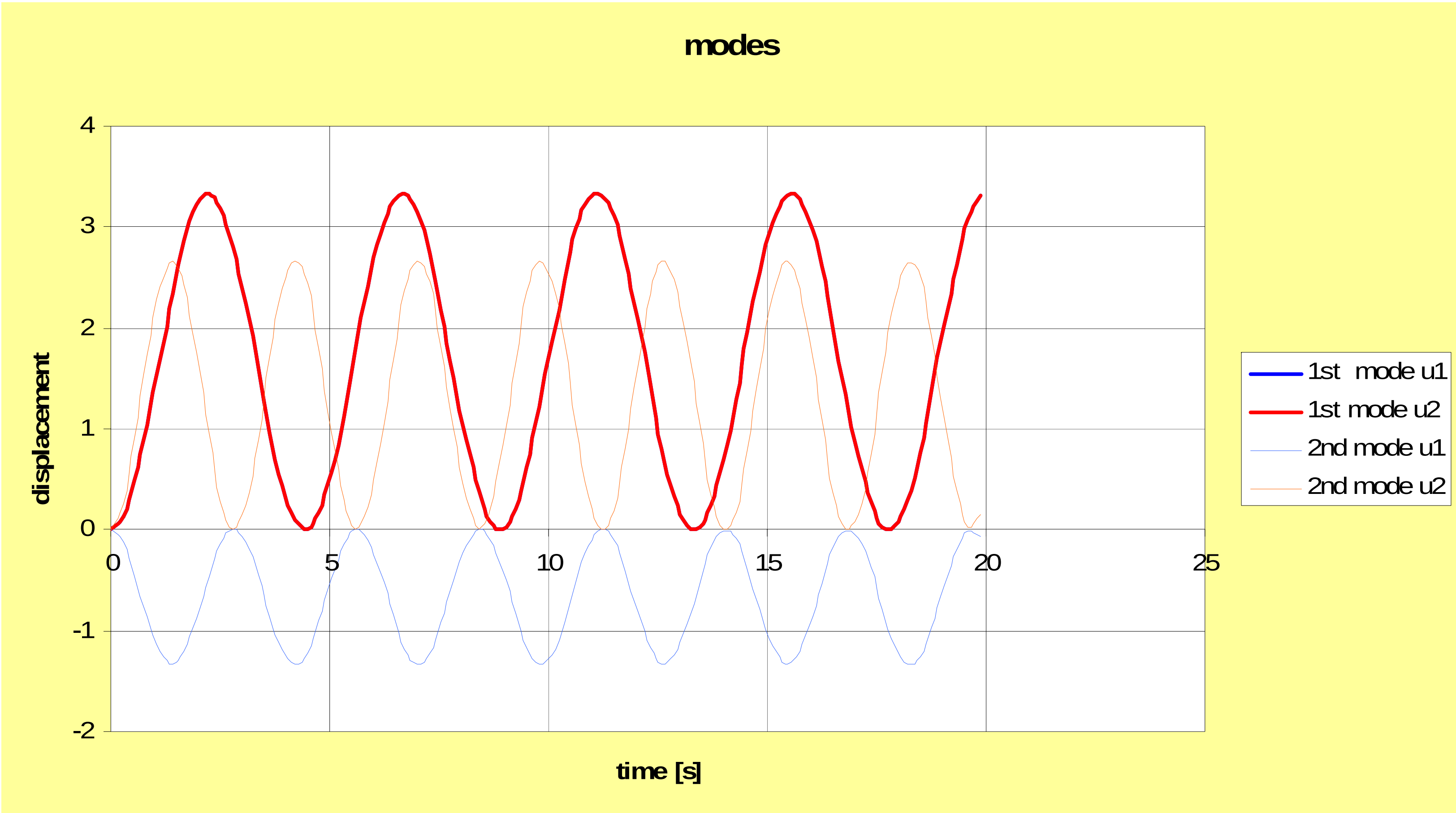
Choosing the right modes for calculation

In general the lowest frequencies and modes are approximated in the best way -> upper bounds for frequencies are found

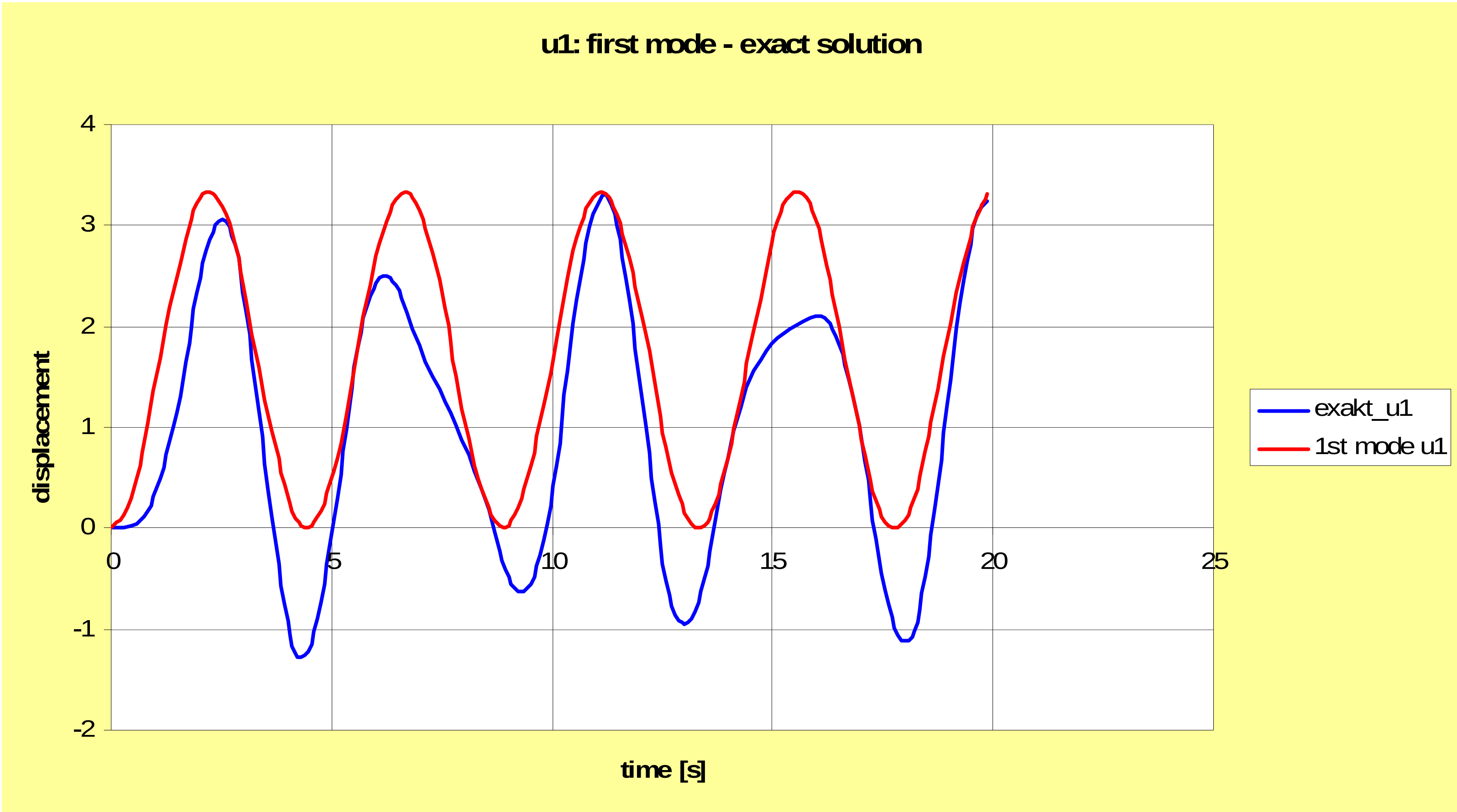
How many and which modes are taken for calculation depends on the problem:

- Earthquake: In some cases only the 10 lowest modes
- Shock: Many modes necessary, $p > 2/3\ n$

Modes in Example 9.7



Only first mode in Example 9.7 for displacement u1



Error Measurement

The accuracy of a solution p < n can be measured...

$$\varepsilon^{p}(t)=\frac{\left\|R(t)-\left[M\ddot{U}^{p}(t)+KU^{p}(t)\right]\right\|_{2}}{\left\|R(t)\right\|_{2}}$$

..and made better by the so called static correction

$$\Delta R = R - \sum_{i=1}^{p} r_i (M\phi_i)$$

$$K \cdot \Delta U(t) = \Delta R(t)$$

Mode Superposition has more advantages then only the reduction of number of necessary operations!

Include damping

- Modal transformation was derived without damping
- Transformation Matrix Φ diagonalizes M and K ....
- ...but not a „free“ chosen damping Matrix C. In this case the equations stay coupled and mode Superposition isn‘t possible

Rayleigh damping

- If C is a linear combination of M and K decoupling is possible, this is called Rayleigh damping

$$C = \alpha \cdot M + \beta \cdot K$$

- In this case, there are only two free parameter for fitting the damping rate

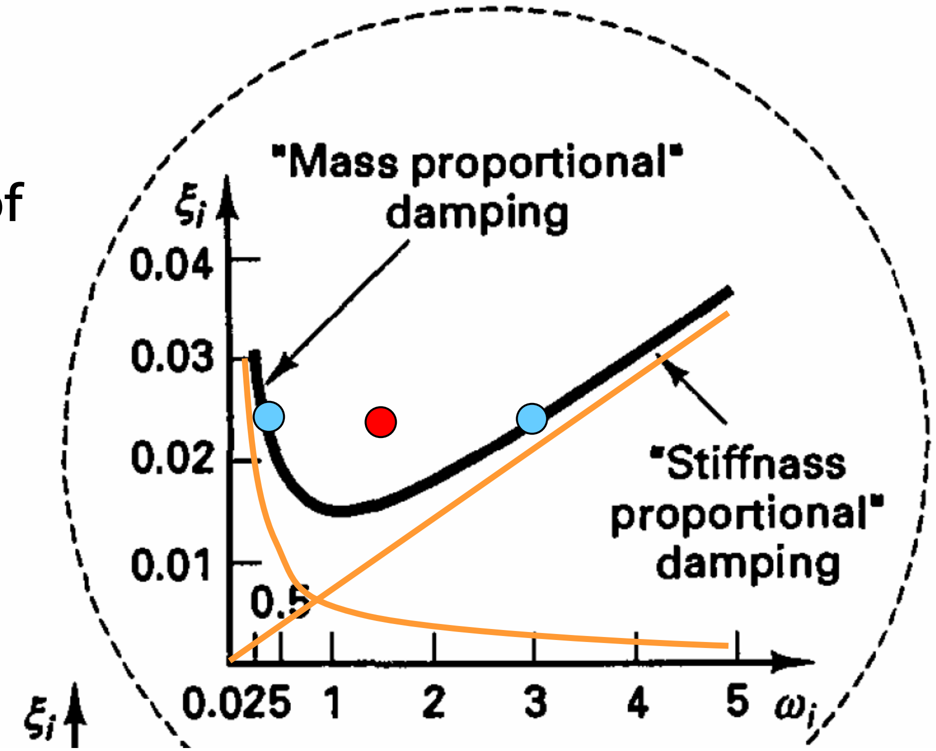
Decoupled equations in case of Rayleigh daamping

$$\ddot{x}_i(t) + 2\omega_i \zeta_i \dot{x}(t) + \omega_i^2 \cdot x_i(t) = r_i(t)$$

$$\zeta_i = \frac{\alpha}{2\omega_i} + \beta\omega_i$$

If an accurate modelling of damping is necessary:

- Direct integration
- Caughey series



Response contributions

Solving a dynamic loaded, damped system, two response contributions can be observed:

- Transient, damped out solution part
- A permanent dynamic response, which is the static response multiplicated by a

  dynamic load factor

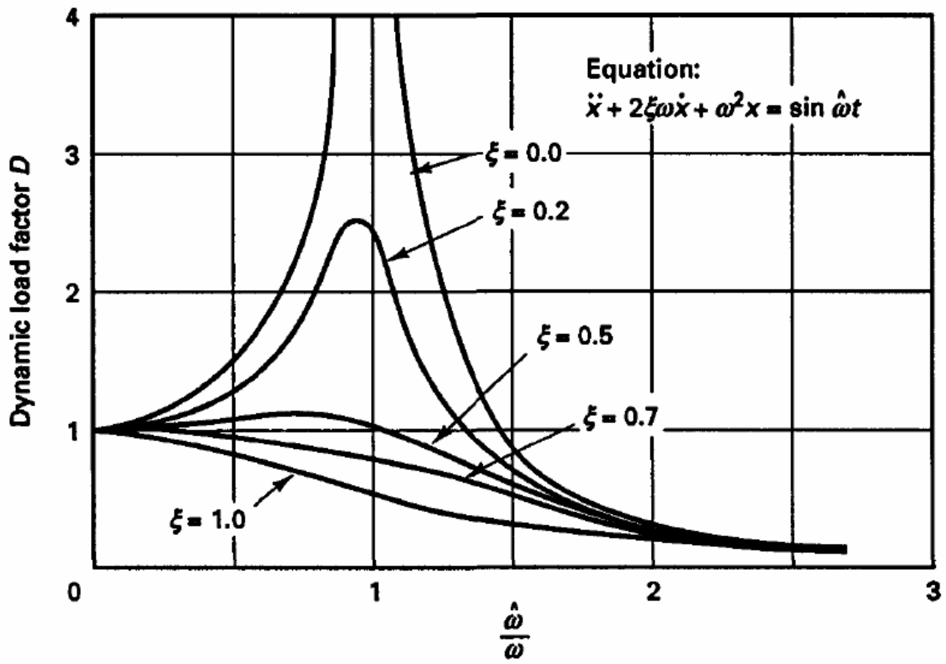


Figure 9.3 The dynamic load factor