The Finite Element Method II

Non-Linear finite element

Use of Constitutive Relations

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Finite element equilibrium equations:

- kinematic variables
- Displacement
- Strain-displacement relations
- Constitutive description

Aims of this lecture:

Present some fundamental material laws in nonlinear finite element analysis

Difference between inelastic analysis and elastic analysis:

- Inelastic - integration of the stresses from state at time *t* to the current strain state is required.

- Elastic - no integration of the stresses is required.

A given strain state can directly deduce:
•Stress
•Tangent stress-strain matrix

The complete nonlinear solution process:

Table 6.8

Three widely used classes of models :

•Elastic model

•Elastoplastic model

•Creep material model

Some basic properties are given in

Table 6.7 Overview of some material descriptions

Elastic material

linear elastic relation: $^{t}\sigma_{ij} = {}^{t}C_{ijrs}\,{}^{t}e_{rs}$

- Constitutive equation:
- ${}^{t}_{0}C_{ijrs}$ is constant elasticity tensor
- The same stress path on unloading as on loading

nolinear elastic relation:

- ${}^{t}_{0}C_{ijrs}$ varies with strain

Elastic material

TL formulation for large deformation:

$$ {}_{0}^{t}S_{ij} = {}_{0}^{t}C_{ijrs} \, {}_{0}^{t}\epsilon_{rs} $$

The fundamental observation:

The second Piola-kirchhoff stress and Green Lagrange strain components do not change measured in a fixed coordinate system when the material is subjected to rigid body motion.

In infinitesimal displacement analysis :

Engineering stress and strain measures
TL formulation is equivalent to Hooke's law

Elastic material

Application in :

•When Stress is small enough

•Before yielding or fracture

Almost all material: Metals, glass, rocks, wood
usually at the beginning of loading

Inelastic material

Difference with elastic materials:

the total stress at time *t* depends on the stress and strain history

Two basic requirements for solution scheme (see table 6.8):

- *Compute the stresses, inelastic strains, and state variable corresponding to the total strains at time* $t+\Delta t$
- *Compute the constitutive relation corresponding to the state evaluated above*

Inelastic material

Three kinds of basic kinematic conditions:

•Small displacement and small strain conditions

•Large displacement and rotations but small strain

•Large displacement and large strain

Large displacement-small strain case represents the material-nonlinear–only case. (geometry-nonlinear , state-nonlinear)

Inelastic material - Elastoplasticity

Classical incremental theory of plastic Based on the Prandtl-Reuss equations

Basic relationship $d\sigma_{ij} = C^{E}_{ijrs}(de_{rs} - de^{P}_{rs})$

Three properties used to calculate the plastic strain

•A yield function: specifies the state of multiaxial stress corresponding to start of plastic flow

•A flow rule: relates the plastic strain increments to the current stress and stress increment

•A hardening rule: specifies how the yield function is modified during plastic flow

Inelastic material - Elastoplasticity

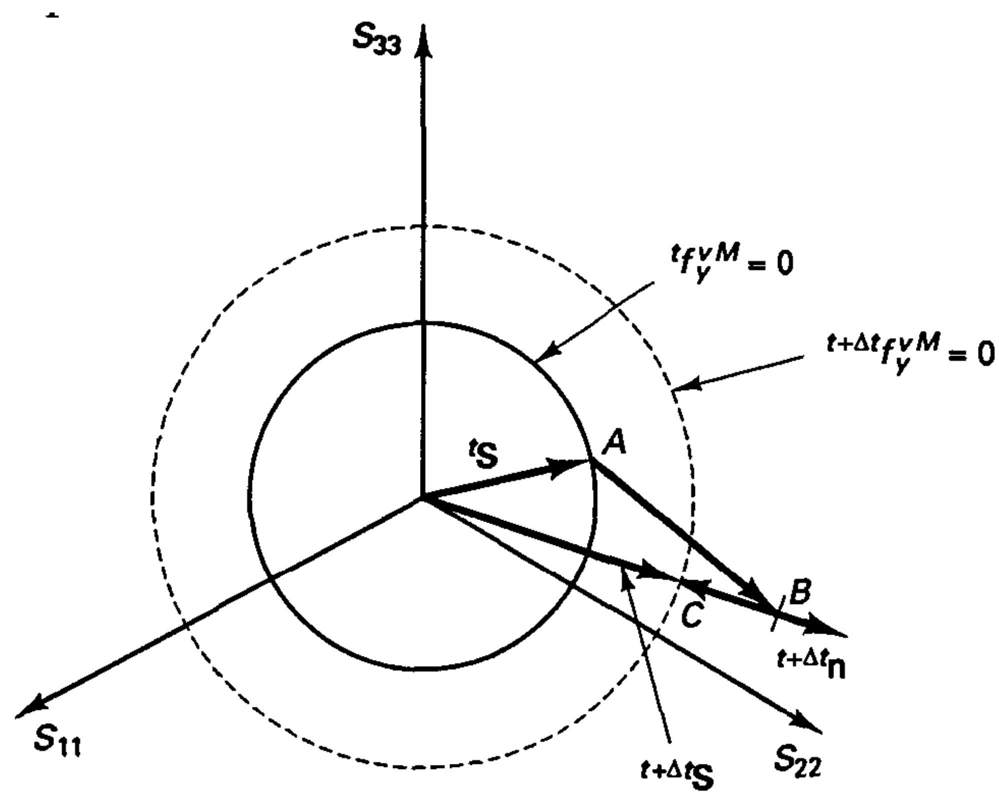
Solution process:

• elastic stress prediction-B

•then judge state- inside or out side the yield surface corresponding to the time *t*

•if lies out side, to make a stress correction C.

Plastic strain increments are instantaneous



Inelastic material - Elastoplasticity

Classical theory :

- Von Mises plasticity with isotropic hardening - metal
- Drucker-Prager - soil and rock

Application in : Metals soils, rocks when subjected to high stress

Inelastic material - Thermoelastoplasticity and creep

Basic rule: $e^{TH} = \alpha\left(\theta - \theta_{ref}\right)$

where e^{TH} is thermal strain, α is the coefficient of thermal expansion and θ_{ref} is the reference temperature.

Constitutive relation(Effective-stress-function algorithm):

$$^{t+\Delta t}\mathbf{S} = \frac{^{t+\Delta t}E}{1 + {}^{t+\Delta t}\nu}\left({}^{t+\Delta t}\mathbf{e}'' - \Delta\mathbf{e}^{P} - \Delta\mathbf{e}^{C}\right)$$

where the known strains are

$$^{t+\Delta t}\mathbf{e}'' = {}^{t+\Delta t}\mathbf{e}' - {}^{t}\mathbf{e}^{P} - {}^{t}\mathbf{e}^{C}$$

Inelastic material - Thermoelastoplasticity and creep

The salient features:

- Young's modulus and Poisson's ratio are temperature-dependent.
- Constant load – strain increase with time lasting
- Constant deformation – stress increases with time lasting
- Creep strain are instantaneous

Application in : Metals and concrete (especially Rolled Compact Concrete) at high temperature

Inelastic material - Viscoplasticity

An important aspect : no yield condition but instead the time-rate effects. The rate of the inelastic response is determined by the instantaneous difference between the effective stress ${}^t\bar{\sigma}$ and the "material" effective stress ${}^t\bar{\sigma}_0$.

Strain increments is decoposed as

$$de_{ij} = de_{ij}^{E} + de_{ij}^{VP}$$

The viscoplastic strain increments at time t are

$$de_{ij}^{VP} = \begin{cases} \beta\phi({}^t\bar{\sigma})\dfrac{3}{2\,{}^t\bar{\sigma}}\,{}^tS_{ij}\,dt & \text{if } {}^t\bar{\sigma} > {}^t\bar{\sigma}_0 \\ 0 & \text{if } {}^t\bar{\sigma} \le {}^t\bar{\sigma}_0 \end{cases}$$

Inelastic material - Viscoplasticity

And

$$\phi({}^t\bar{\sigma}) = \left(\frac{{}^t\bar{\sigma} - {}^t\bar{\sigma}_0}{{}^t\bar{\sigma}_0}\right)^N$$

where,

${}^t\bar{\sigma}_0$ is corresponding to the accumulated effective viscoplatic strain; β is a material constant with unit 1/time; N is another material constant.

Viscoplastic strains are accumulated as longas the effetive stress is larger than the material effetive stress. And this accumulation is determined by the above two constants.

Inelastic material - Viscoplasticity

Advantage of viscoplastic model:

Various functional dependencies can be directely and easily included in the caculations.

Reason:

there is no explicit yield condition.

Method:

Integrating the inelastic strains until the effective stress is equal to the material effetive stress given by the effective viscoplastic strain.

Large strain Elastoplasticity

Different measures Contrast:

•Small strain - Total strain – numerical integration only in the caculation of the inelastic strain from time t to time $t + \Delta t$.

•Large strain - Rate–type formulation – numerical integration in rates of stress, strain ,and rotational effects, which leads to additional numerical errors, so requires significantly smaller solution steps.

•Rate –type formulation are based on the Jaumann stress rate-velocity strain description (example 6.24)

Large strain Elastoplasticity

One important feature: the uniaxial stress-strain law is given by Cauchy stress-logarithmic strain relationship

By using Cauchy stress-logarithmic strain relationship, the basic equation for infinitesimal strain problem can be generalized to large strain.

The Large strain formulation is a direct extension of the formulation used for small strain. (Table 6.10 reduces)

Example 6.24

It demonstrates the differences when using different stress-strain measures with the same material constants.

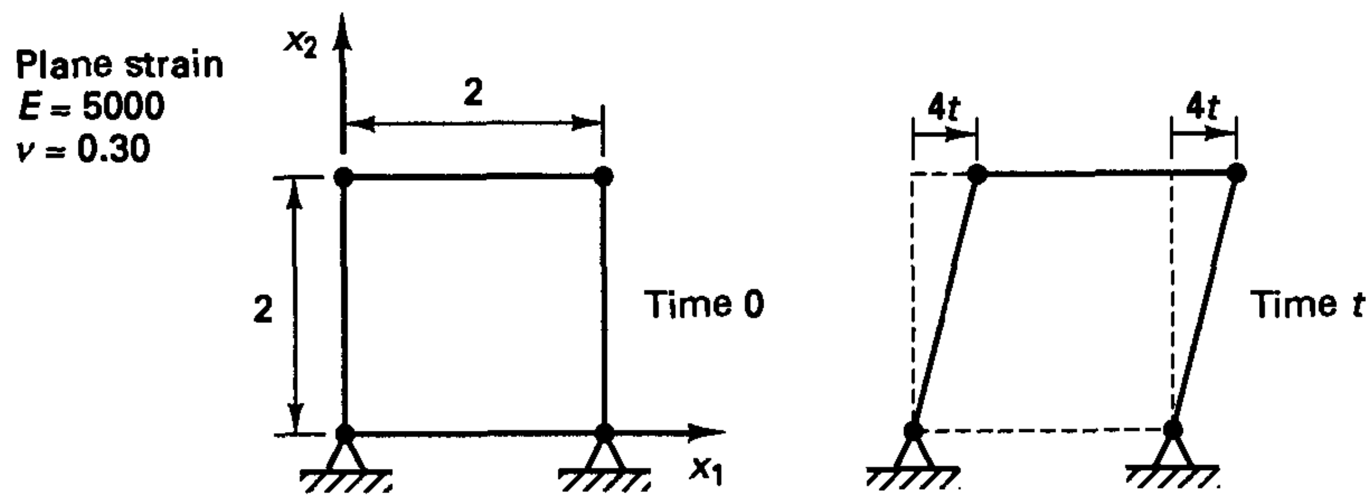
Four-node element

Given the displacements - a function of time

Calculate the Cauchy stress using:

- The total fomulation
- The rate formulation

Example 6.24



i:for total fomulation

$$ {}_0^t S_{ij} = {}_0^t C_{ijrs} \, {}_0^t \epsilon_{rs} $$

Example 6.24 i

The nodal coordinates at time 0 and *t* are separately,

$$ {}^{0}x_1^1 = 2; \quad {}^{0}x_2^1 = 2; \quad {}^{0}x_1^2 = 0; \quad {}^{0}x_2^2 = 2; $$

$$ {}^{0}x_1^3 = 0; \quad {}^{0}x_2^3 = 0; \quad {}^{0}x_1^4 = 2; \quad {}^{0}x_2^4 = 0; $$

$$ {}^{t}x_1^1 = 2 + 4t; \quad {}^{t}x_2^1 = 2; \quad {}^{t}x_1^2 = 4t; \quad {}^{t}x_2^2 = 2; $$

$$ {}^{t}x_1^3 = 0; \quad {}^{t}x_2^3 = 0; \quad {}^{t}x_1^4 = 2; \quad {}^{t}x_2^4 = 0; $$

Substitute them into following expressions:

Example 6.24 i

$$h_1 = \tfrac{1}{4}(1 + {}^0x_1)(1 + {}^0x_2); \qquad h_2 = \tfrac{1}{4}(1 - {}^0x_1)(1 + {}^0x_2)$$

$$h_3 = \tfrac{1}{4}(1 - {}^0x_1)(1 - {}^0x_2); \qquad h_4 = \tfrac{1}{4}(1 + {}^0x_1)(1 - {}^0x_2)$$

$$\frac{\partial h_1}{\partial {}^0x_1} = \frac{1}{4}(1 + {}^0x_2); \qquad \frac{\partial h_2}{\partial {}^0x_1} = -\frac{1}{4}(1 + {}^0x_2)$$

$$\frac{\partial h_3}{\partial {}^0x_1} = -\frac{1}{4}(1 - {}^0x_2); \qquad \frac{\partial h_4}{\partial {}^0x_1} = \frac{1}{4}(1 - {}^0x_2)$$

$$\frac{\partial h_1}{\partial {}^0x_2} = \frac{1}{4}(1 + {}^0x_1); \qquad \frac{\partial h_2}{\partial {}^0x_2} = \frac{1}{4}(1 - {}^0x_1)$$

$$\frac{\partial h_3}{\partial {}^0x_2} = -\frac{1}{4}(1 - {}^0x_1); \qquad \frac{\partial h_4}{\partial {}^0x_2} = -\frac{1}{4}(1 + {}^0x_1)$$

$$\frac{\partial {}^t x_i}{\partial {}^0 x_j} = \sum_{k=1}^{4} \left(\frac{\partial h_k}{\partial {}^0 x_j}\right) {}^t x_i^k$$

Example 6.24 i

So the deformation gradient is (referrence Example6.6)

$$ {}_0^t\mathbf{X} = \begin{bmatrix} 1 & 2t \\ 0 & 1 \end{bmatrix} $$

Then we get the Green-Lagrange strain tensor by expression

$$ {}_0^t\varepsilon = \frac{1}{2}({}_0^tX^T \, {}_0^tX - I) \qquad (6.54) $$

$$ {}_0^t\boldsymbol{\epsilon} = \begin{bmatrix} 0 & t \\ t & 2t^2 \end{bmatrix} $$

Example 6.24 i

Using the material matrix C of Plane strain in Table4.3

$$\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1 & \frac{\nu}{1-\nu} & 0 \\ \frac{\nu}{1-\nu} & 1 & 0 \\ 0 & 0 & \frac{1-2\nu}{2(1-\nu)} \end{bmatrix}$$

And with the given values of E=5000 and v=0.3,we obtain as nonzero values:

$$C_{1111} = 6731, C_{2211} = C_{1122} = 2885, C_{2222} = 6731, C_{1212} = 1923$$

Example 6.24 i

Subtitude then into equation (a) $\quad {}_0^t S_{ij} = {}_0^t C_{ijrs} \, {}_0^t \epsilon_{rs}$
We have

$$ {}_0^t S_{11} = {}_0^t C_{1111} \cdot {}_0^t \varepsilon_{11} + {}_0^t C_{1122} \cdot {}_0^t \varepsilon_{22} = 6731 \cdot 0 + 2885 \cdot 2t^2 = 5770t^2 $$

$$ {}_0^t S_{12} = {}_0^t C_{1212} \cdot {}_0^t \varepsilon_{12} + {}_0^t C_{1221} \cdot {}_0^t \varepsilon_{21} = 1923 \cdot t + 1923 \cdot t = 3846t $$

$$ {}_0^t S_{22} = 13462t^2 $$

And using the standard transformationa between the second Piola-kirchhoff and Cauchy stresses equation

$$ {}^t\tau_{mn} = \frac{{}^t\rho}{{}^0\rho} \, {}_0^t x_{m,i} \, {}_0^t x_{n,j} \, {}_0^t S_{ij} \qquad (6.69) $$

Example 6.24 i

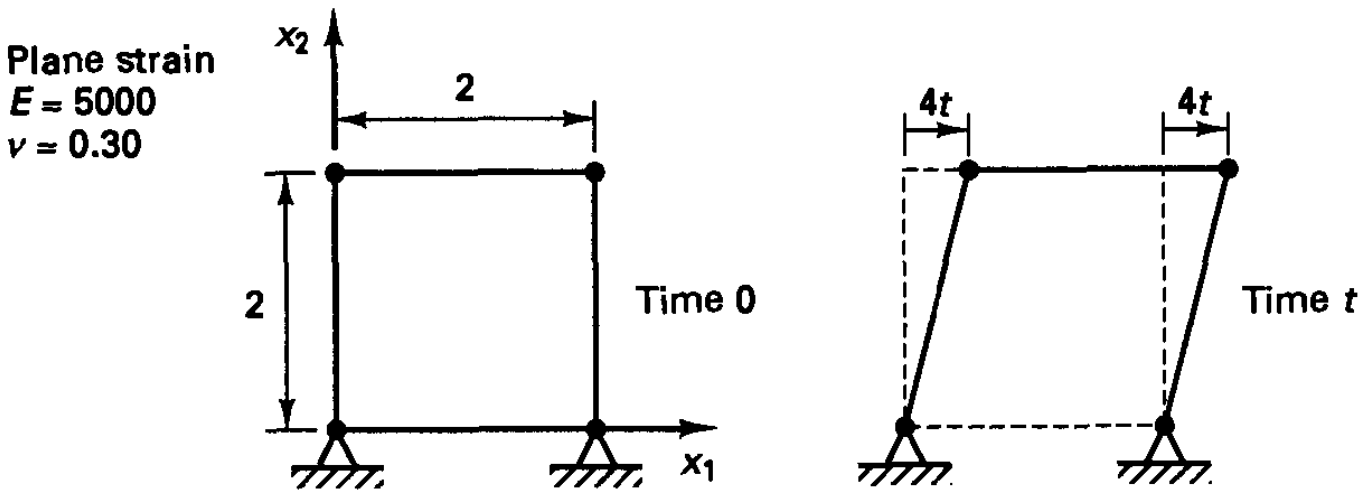
Where $$\frac{\rho_0}{\rho_t} = \det({}_0^t X) = 1$$

So

$$
\begin{aligned}
{}_0^t\tau_{11} &= {}_0^t x_{1,1} \cdot {}_0^t x_{1,1} \cdot {}_0^t S_{11} + {}_0^t x_{1,2} \cdot {}_0^t x_{1,2} \cdot {}_0^t S_{22} + {}_0^t x_{1,1} \cdot {}_0^t x_{1,2} \cdot {}_0^t S_{12} + {}_0^t x_{1,2} \cdot {}_0^t x_{1,1} \cdot {}_0^t S_{21} \\
&= 1 \cdot 5770t^2 + 2t \cdot 2t \cdot 13462t^2 + 2t \cdot 3846t + 2t \cdot 3846t \\
&= 21154t^2 + 53848t^4 \\
{}_0^t\tau_{22} &= 13462t^2 \\
{}_0^t\tau_{12} &= 3846t + 26924t^3
\end{aligned}
$$

(c)

Example 6.24 ii



ii:for rate fomulation

$$ {}^{t}\overset{\nabla}{\tau}_{ij} = {}_{t}C_{ijrs}\, {}^{t}D_{rs} $$

Example 6.24 ii

By expression

$$\mathbf{L} = \dot{\mathbf{X}}\mathbf{X}^{-1} \qquad (6.40)$$

We get

$${}^{t}\mathbf{L} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$$

The velocity strain strain tensor is computed as given in

$$\mathbf{D} = \tfrac{1}{2}\mathbf{R}(\dot{\mathbf{U}}\mathbf{U}^{-1} + \mathbf{U}^{-1}\dot{\mathbf{U}})\mathbf{R}^{T} \qquad (6.42)$$

Hence,

$${}^{t}\mathbf{D} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \qquad {}^{t}\mathbf{W} = \begin{bmatrix} 0 & +1 \\ -1 & 0 \end{bmatrix}$$

Example 6.24 ii

Now we use the same constitutive matrix C to substitude into

$$ {}^{t}\overset{\nabla}{\tau}_{ij} = {}_{t}C_{ijrs}\, {}^{t}D_{rs} \qquad \text{(b)} $$

Then we get

$$ {}^{t}\overset{\nabla}{\tau}_{11} = {}_{t}C_{11rs} \cdot {}^{t}D_{rs} = {}_{t}C_{1111} \cdot {}^{t}D_{11} + {}_{t}C_{1112} \cdot {}^{t}_{0}D_{12} + {}_{t}C_{1121} \cdot {}^{t}D_{21} + {}_{t}C_{1122} \cdot {}^{t}D_{22} = 0 $$

$$ {}^{t}\overset{\nabla}{\tau}_{22} = 0 $$

$$ {}^{t}\overset{\nabla}{\tau}_{12} = 3846 $$

Example 6.24 ii

Then put them into

$$ {}^{t}\overset{\nabla}{\tau}_{ij} = {}^{t}\dot{\tau}_{ij} + {}^{t}\tau_{ip}\,{}^{t}W_{pj} + {}^{t}\tau_{jp}\,{}^{t}W_{pi} \qquad \text{(c)} $$

Then we get Jaumann stress rate as following

$$ {}^{t}\overset{\nabla}{\tau}_{11} = {}^{t}\dot{\tau}_{11} \cdot -2\,{}^{t}\,{}^{t}\tau_{11} = 0 $$

$$ {}^{t}\overset{\nabla}{\tau}_{22} = {}^{t}\dot{\tau}_{22} + {}^{t}\tau_{21} + {}^{t}\tau_{21} = 0 $$

$$ {}^{t}\overset{\nabla}{\tau}_{12} = {}^{t}\dot{\tau}_{12} + {}^{t}\tau_{11} - {}^{t}\tau_{22} = 3846 $$

Example 6.24 ii

Reorganize the above equations we get the time rates of Cauchy stress

$$\begin{bmatrix} {}^t\dot{\tau}_{11} \\ {}^t\dot{\tau}_{22} \\ {}^t\dot{\tau}_{12} \end{bmatrix} = \begin{bmatrix} 2\ {}^t\tau_{12} \\ -2\ {}^t\tau_{12} \\ 3846 + {}^t\tau_{22} - {}^t\tau_{11} \end{bmatrix}$$

Soveing the differenctial equations to obtain

$$\begin{bmatrix} {}^t\tau_{11} \\ {}^t\tau_{22} \\ {}^t\tau_{12} \end{bmatrix} = \begin{bmatrix} 1900(1 - \cos 2t) \\ -1900(1 - \cos 2t) \\ 1900 \sin 2t \end{bmatrix} \qquad \text{(d)}$$

Example 6.24

Comparing:

$$\begin{bmatrix} {}^t\tau_{11} \\ {}^t\tau_{22} \\ {}^t\tau_{12} \end{bmatrix} = \begin{bmatrix} 21{,}000t^2 + 54{,}000t^4 \\ 13{,}000t^2 \\ 3800t + 27{,}000t^3 \end{bmatrix} \qquad \begin{bmatrix} {}^t\tau_{11} \\ {}^t\tau_{22} \\ {}^t\tau_{12} \end{bmatrix} = \begin{bmatrix} 1900(1 - \cos 2t) \\ -1900(1 - \cos 2t) \\ 1900 \sin 2t \end{bmatrix}$$

- (c) And (d) are quite different when t is larger than 0.1
- The normal stresses are generated as zero when infinitesimal small strain are assumed
- (d) has period π

Thank you!