Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Structural Elements

Beam and Axis-symmetric Shell Elements

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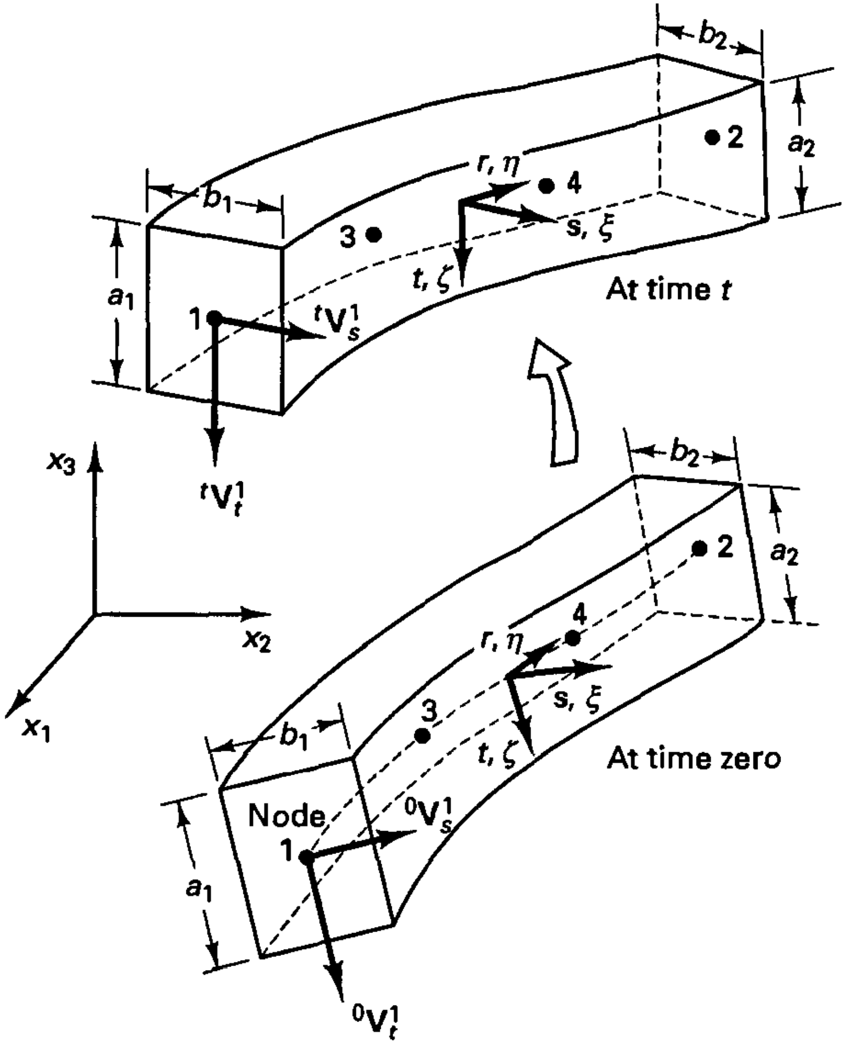
Beam element



assumptions:

- we neglect shear effects (Bernoulli beam theory)

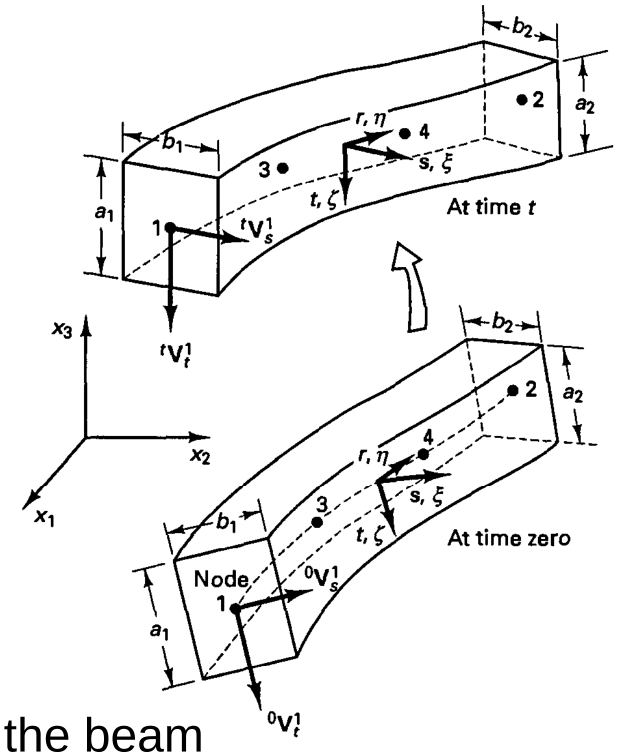
→ plane sections initially normal to the neutral axis remain plane

- large displacements, large rotations, but small strains

→ cross-sectional area does not change

Beam element

Geometry:



$$ {}^t x_i = \sum_{k=1}^{q} h_k \, {}^t x_i^k + \frac{t}{2}\sum_{k=1}^{q} a_k h_k \, {}^t V_{ti}^k + \frac{s}{2}\sum_{k=1}^{q} b_k h_k \, {}^t V_{si}^k $$

i = 1, 2, 3

$ {}^t x_1, {}^t x_2, {}^t x_3 $ – coordinates of a typical point in the beam

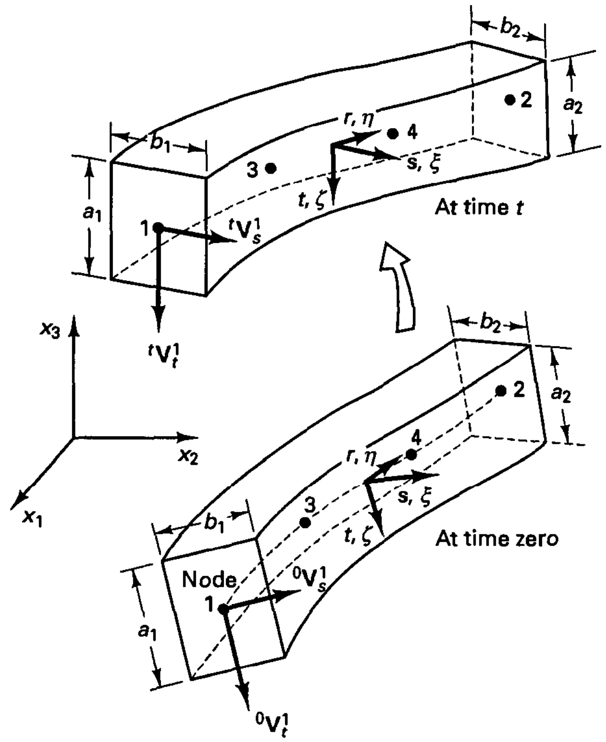
h_k – interpolation functions

a_k, b_k – cross-sectional dimensions

$ {}^t V_t^k, {}^t V_s^k $ – components of unit vector in direction s or t at nodal point k at time t → director vector

Beam element

Displacement components:



at time t:

$$^{t}u_i = {}^{t}x_i - {}^{0}x_i$$

increment from time t to t+Δt:

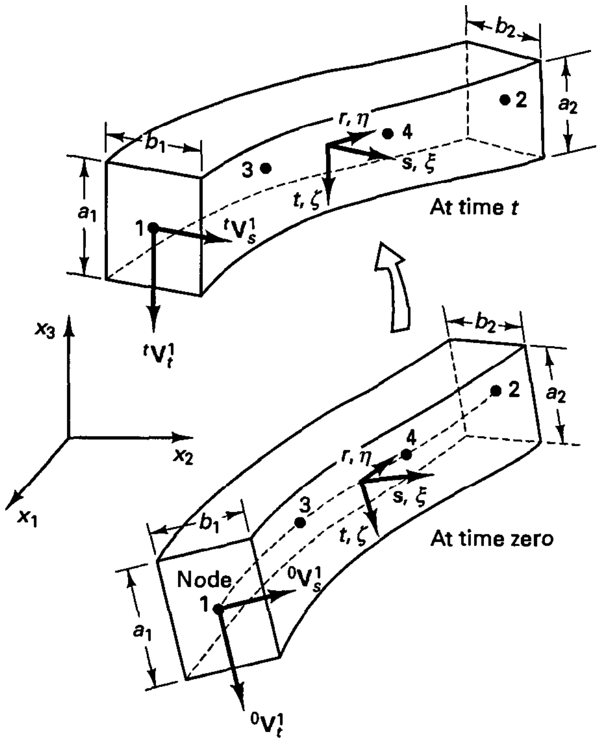
$$u_i = {}^{t+\Delta t}x_i - {}^{t}x_i$$

Beam element

$$ {}^{t}x_i = \sum_{k=1}^{q} h_k\, {}^{t}x_i^k + \frac{t}{2}\sum_{k=1}^{q} a_k h_k\, {}^{t}V_{ti}^k + \frac{s}{2}\sum_{k=1}^{q} b_k h_k\, {}^{t}V_{si}^k $$

$$ {}^{t}u_i = {}^{t}x_i - {}^{0}x_i $$

$$ u_i = {}^{t+\Delta t}x_i - {}^{t}x_i $$

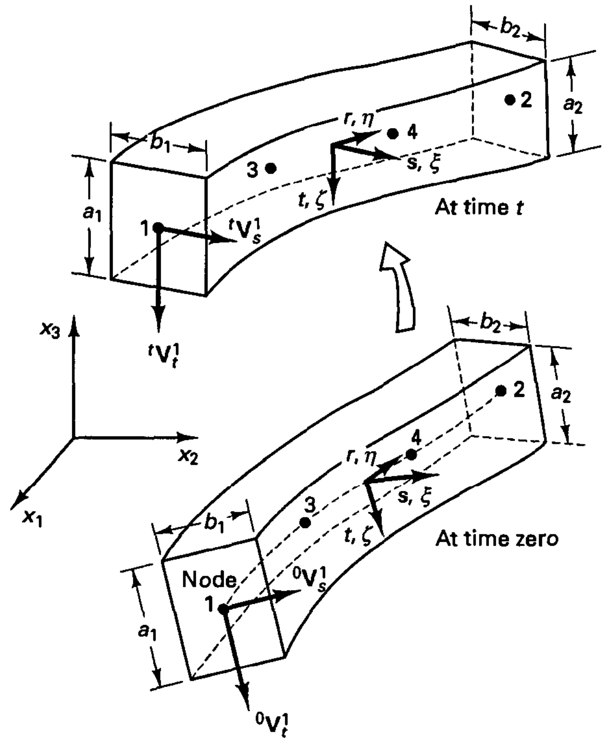


displacement components in terms of the nodal point displacements and changes in the direction cosines of the nodal point director vectors:

$$ {}^{t}u_i = \sum_{k=1}^{q} h_k\, {}^{t}u_i^k + \frac{t}{2}\sum_{k=1}^{q} a_k h_k \left({}^{t}V_{ti}^k - {}^{0}V_{ti}^k\right) + \frac{s}{2}\sum_{k=1}^{q} b_k h_k \left({}^{t}V_{si}^k - {}^{0}V_{si}^k\right) $$

$$ u_i = \sum_{k=1}^{q} h_k u_i^k + \frac{t}{2}\sum_{k=1}^{q} a_k h_k V_{ti}^k + \frac{s}{2}\sum_{k=1}^{q} b_k h_k V_{si}^k $$

Beam element



with the director vector increments:

$$V_{ti}^{k} = {}^{t+\Delta t}V_{ti}^{k} - {}^{t}V_{ti}^{k}$$

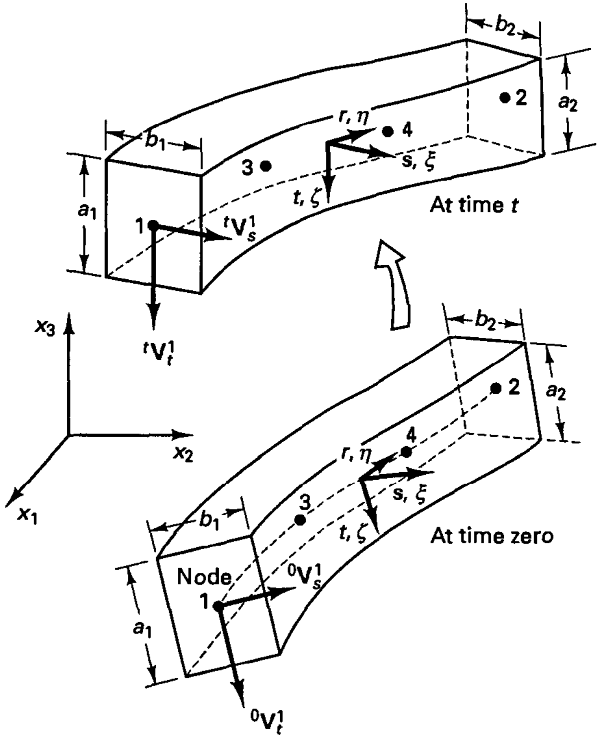
$$V_{si}^{k} = {}^{t+\Delta t}V_{si}^{k} - {}^{t}V_{si}^{k}$$

$${}^{t}u_i = \sum_{k=1}^{q} h_k\, {}^{t}u_i^k + \frac{t}{2}\sum_{k=1}^{q} a_k h_k \left({}^{t}V_{ti}^{k} - {}^{0}V_{ti}^{k}\right) + \frac{s}{2}\sum_{k=1}^{q} b_k h_k \left({}^{t}V_{si}^{k} - {}^{0}V_{si}^{k}\right)$$

$$u_i = \sum_{k=1}^{q} h_k u_i^k + \frac{t}{2}\sum_{k=1}^{q} a_k h_k V_{ti}^{k} + \frac{s}{2}\sum_{k=1}^{q} b_k h_k V_{si}^{k}$$

Beam element

For a full linearisation of the principle of virtual work we define the vector of nodal rotational degrees of freedom θ_k:



$$V_t^k = \theta_k \times {}^tV_t^k + \frac{1}{2}\theta_k \times \left(\theta_k \times {}^tV_t^k\right)$$

$$V_s^k = \theta_k \times {}^tV_s^k + \frac{1}{2}\theta_k \times \left(\theta_k \times {}^tV_s^k\right)$$

The purpose is to evaluate (approximations to) the new director vectors.

Beam element

linearised equilibrium equations:

in TL-formulation

$$\int_{^0V} {}_0C_{ijrs}\,{}_0e_{rs}\,\delta_0 e_{ij}\,d\,{}^0V + \int_{^0V} {}_0^tS_{ij}\,{}_0\delta_0\eta_{ij}\,d\,{}^0V = {}^{t+\Delta t}R - \boxed{\int_{^0V} {}_0^tS_{ij}\,\delta_0 e_{ij}\,d\,{}^0V}$$

in UL formulation

$$\int_{^tV} {}_tC_{ijrs}\,{}_te_{rs}\,\delta_t e_{ij}\,d\,{}^tV + \int_{^0V} {}^t\tau_{ij}\,\delta_t\eta_{ij}\,d\,{}^tV = {}^{t+\Delta t}R - \boxed{\int_{^tV} {}^t\tau_{ij}\,\delta_t e_{ij}\,d\,{}^tV}$$

linear strain increments now include *quadratic* terms in rotations

Beam element

Finite-element equations:

$$ {}^{t}K \begin{bmatrix} \vdots \\ u_k \\ \theta_k \\ \vdots \end{bmatrix} = {}^{t+\Delta t}R - {}^{t}F $$

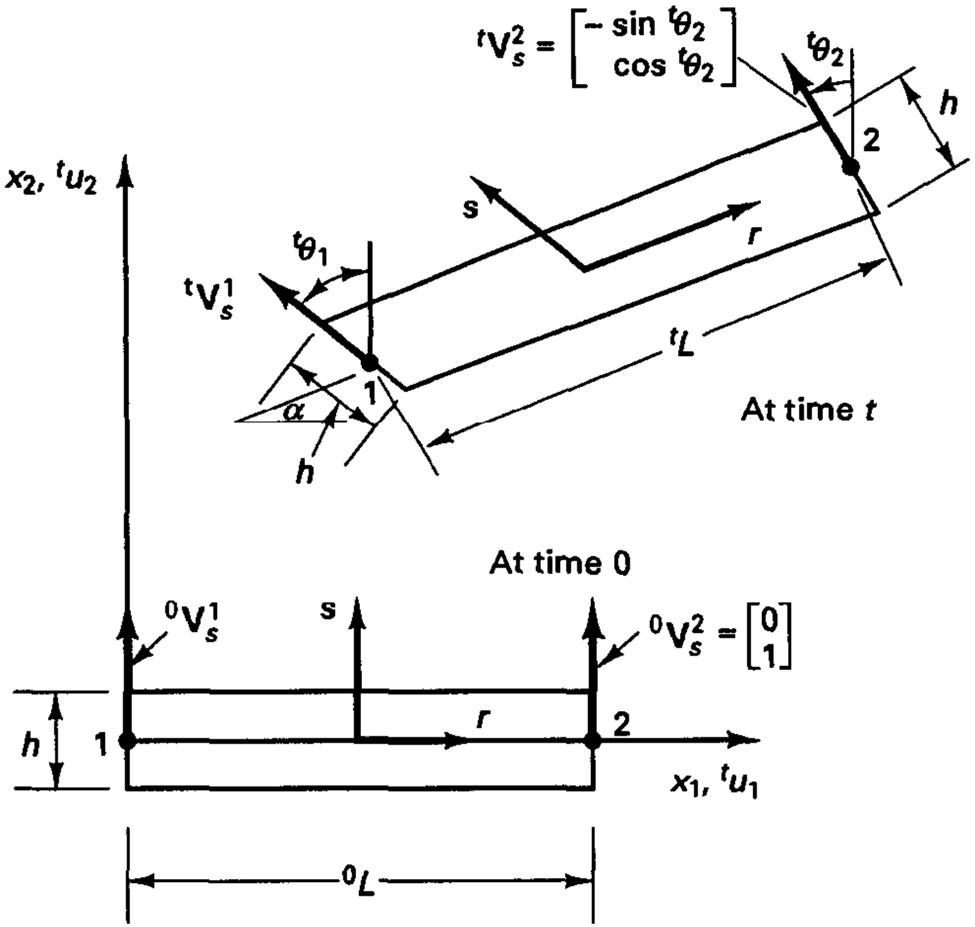
solving for u_k and θ_k, we obtain approximations for the nodal point displacements and director vectors at time t+Δt using

$$ {}^{t+\Delta t}u_k = {}^{t}u_k + u_k $$

$$ {}^{t+\Delta t}V_t^k = {}^{t}V_t^k + \int_{\theta_k} d\theta_k \times {}^{\tau}V_t^k $$

$$ {}^{t+\Delta t}V_s^k = {}^{t}V_s^k + \int_{\theta_k} d\theta_k \times {}^{\tau}V_s^k $$

Example



2-node beam element

We are looking for the coordinate and displacement interpolations and derivatives that are reqired for the calculations of the stress.displacement matrices.

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Example

Geometry:

$$ {}^{t}x_i = \sum_{k=1}^{q} h_k \, {}^{t}x_i^k + \frac{s}{2} \sum_{k=1}^{q} b_k h_k \, {}^{t}V_{si}^k $$

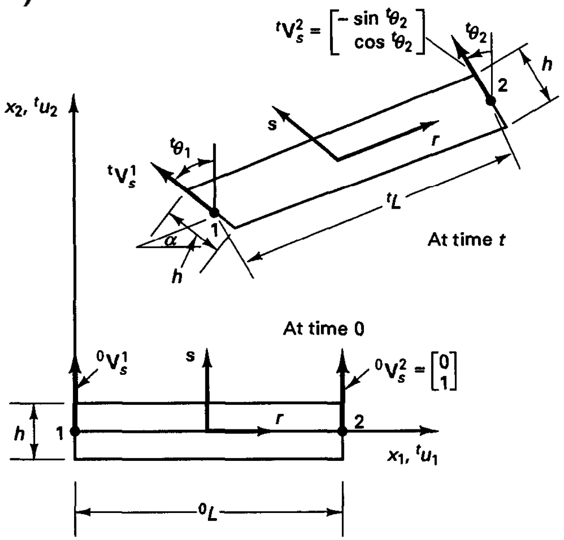
$$ h_1 = \frac{1-r}{2}, h_2 = \frac{1+r}{2} $$

$$ {}^{t}x_1 = \left(\frac{1-r}{2}\right) {}^{t}x_1^1 + \left(\frac{1+r}{2}\right) {}^{t}x_1^2 - \frac{sh}{2}\left(\frac{1-r}{2}\right) \sin {}^{t}\theta_1 - \frac{sh}{2}\left(\frac{1+r}{2}\right) \sin {}^{t}\theta_2 $$

$$ {}^{t}x_2 = \left(\frac{1-r}{2}\right) {}^{t}x_2^1 + \left(\frac{1+r}{2}\right) {}^{t}x_2^2 + \frac{sh}{2}\left(\frac{1-r}{2}\right) \cos {}^{t}\theta_1 + \frac{sh}{2}\left(\frac{1+r}{2}\right) \cos {}^{t}\theta_2 $$

$$ {}^{0}x_1 = \left(\frac{1+r}{2}\right) {}^{0}L $$

$$ {}^{0}x_2 = \frac{sh}{2} $$



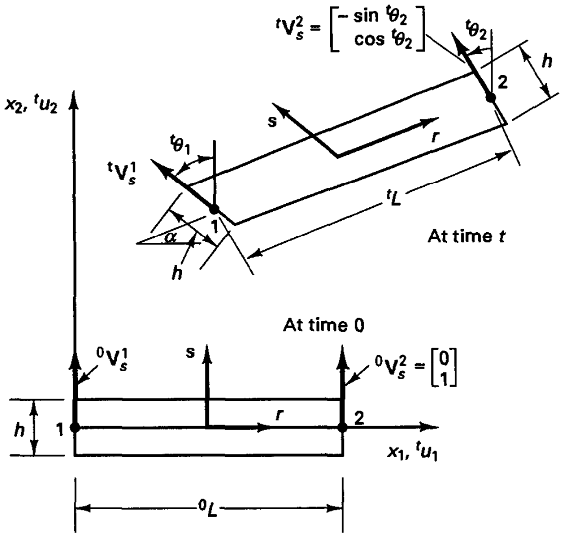
Example

Displacement components at time t:

$$ {}^{t}u_i = {}^{t}x_i - {}^{0}x_i $$

$$ {}^{t}u_1 = \left(\frac{{}^{t}x_1^1 + {}^{t}x_1^2 - {}^{0}L}{2}\right) + \left(\frac{{}^{t}x_1^2 - {}^{t}x_1^1 - {}^{0}L}{2}\right)r - \frac{sh}{2}\left[\left(\frac{1-r}{2}\right)\sin {}^{t}\theta_1 + \left(\frac{1+r}{2}\right)\sin {}^{t}\theta_2\right] $$

$$ {}^{t}u_2 = \left(\frac{{}^{t}x_2^1 + {}^{t}x_2^2}{2}\right) + \left(\frac{{}^{t}x_2^2 - {}^{t}x_2^1}{2}\right)r + \frac{sh}{2}\left[\left(\frac{1-r}{2}\right)\cos {}^{t}\theta_1 + \left(\frac{1+r}{2}\right)\cos {}^{t}\theta_2 - 1\right] $$



Example

Incremental displacements:

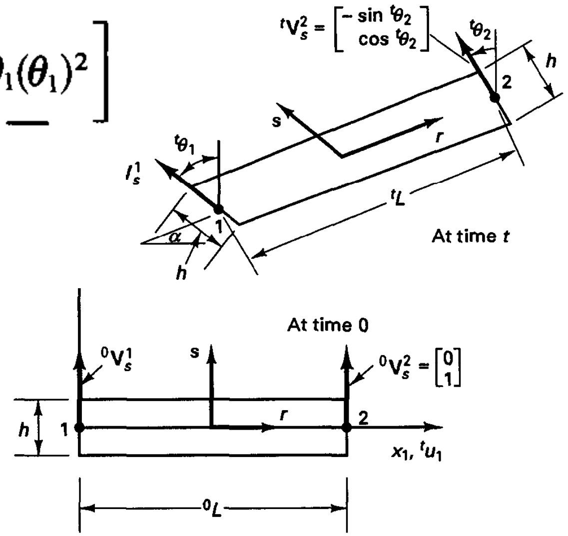
$$u_i = \sum_{k=1}^{q} h_k u_i^k + \frac{s}{2} \sum_{k=1}^{q} b_k h_k V_{si}^k \qquad V_{si}^k = {}^{t+\Delta t}V_{si}^k - {}^{t}V_{si}^k$$

$$u_1 = \frac{1-r}{2} u_1^1 + \frac{1+r}{2} u_1^2 + \frac{sh}{2}\left(\frac{1-r}{2}\right)\left[(-\cos {}^t\theta_1)\theta_1 + \frac{1}{2}\sin {}^t\theta_1(\theta_1)^2\right]$$

$$+ \frac{sh}{2}\left(\frac{1+r}{2}\right)\left[(-\cos {}^t\theta_2)\theta_2 + \frac{1}{2}\sin {}^t\theta_2(\theta_2)^2\right]$$

$$u_2 = \frac{1-r}{2} u_2^1 + \frac{1+r}{2} u_2^2 + \frac{sh}{2}\left(\frac{1-r}{2}\right)\left[(-\sin {}^t\theta_1)\theta_1 - \frac{1}{2}\cos {}^t\theta_1(\theta_1)^2\right]$$

$$+ \frac{sh}{2}\left(\frac{1+r}{2}\right)\left[(-\sin {}^t\theta_2)\theta_2 - \frac{1}{2}\cos {}^t\theta_2(\theta_2)^2\right]$$



Example

Derivatives of the Jacobian matrix in UL formulation:

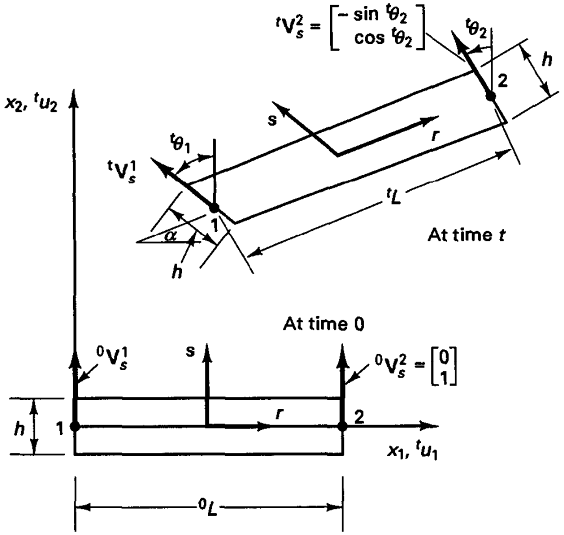
$$\frac{\partial\, {}^t x_1}{\partial r} = \frac{L\cos\alpha}{2} - \frac{sh}{4}\left(\sin {}^t\theta_2 - \sin {}^t\theta_1\right)$$

$$\frac{\partial\, {}^t x_1}{\partial s} = \left(-\frac{h}{2}\right)\left[\left(\frac{1-r}{2}\right)\sin {}^t\theta_1 + \left(\frac{1+r}{2}\right)\sin {}^t\theta_2\right]$$

$$\frac{\partial\, {}^t x_2}{\partial r} = \frac{L\sin\alpha}{2} + \frac{sh}{4}\left(\cos {}^t\theta_2 - \cos {}^t\theta_1\right)$$

$$\frac{\partial\, {}^t x_2}{\partial s} = \frac{h}{2}\left[\left(\frac{1-r}{2}\right)\cos {}^t\theta_1 + \left(\frac{1+r}{2}\right)\cos {}^t\theta_2\right]$$

We assumed ${}^tL = {}^0L = L$



Example

Jacobian matrix in TL formulation:

$$ {}^{0}\mathbf{J} = \begin{bmatrix} \dfrac{{}^{0}L}{2} & 0 \\ 0 & \dfrac{h}{2} \end{bmatrix} $$

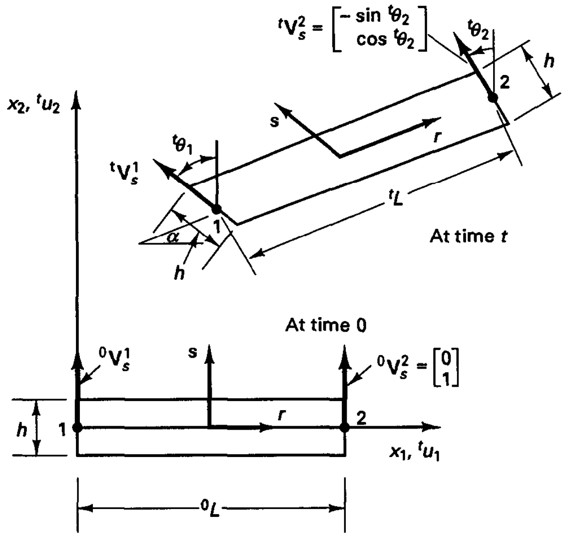
derivatives of the displacements:

$$ {}^{t}_{0}u_{1,1} = (\cos\alpha - 1) - \frac{sh}{2L}(\sin{}^{t}\theta_2 - \sin{}^{t}\theta_1) $$

$$ {}^{t}_{0}u_{1,2} = -\left(\frac{1-r}{2}\right)\sin{}^{t}\theta_1 - \left(\frac{1+r}{2}\right)\sin{}^{t}\theta_2 $$

$$ {}^{t}_{0}u_{2,1} = \sin\alpha + \frac{sh}{2L}(\cos{}^{t}\theta_2 - \cos{}^{t}\theta_1) $$

$$ {}^{t}_{0}u_{2,2} = \left(\frac{1-r}{2}\right)\cos{}^{t}\theta_1 + \left(\frac{1+r}{2}\right)\cos{}^{t}\theta_2 - 1 $$



We again assumed ${}^{t}L = {}^{0}L = L$