More-dimensional Elements in nonlinear Analysis

- *2D: Axisymmetric, Plane Strain and Plane Stress Elements*
- *3D: Solid Elements*

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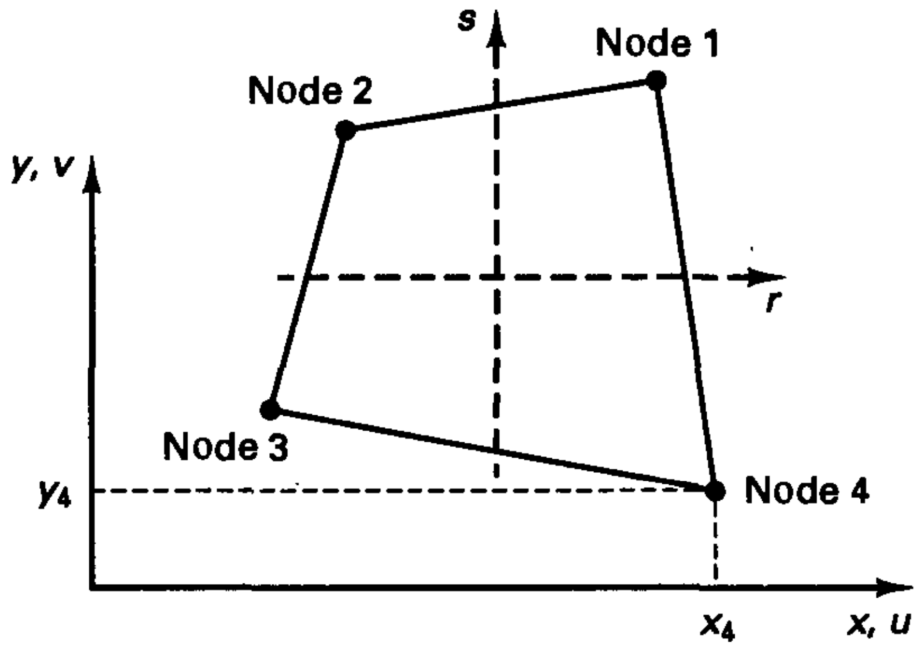


4-node, 2D-Element

Problem	Displacement components	Strain vector ϵ^T	Stress vector τ^T
Plane stress	u, v	$[\epsilon_{xx}\ \epsilon_{yy}\ \gamma_{xy}]$	$[\tau_{xx}\ \tau_{yy}\ \tau_{xy}]$
Plane strain	u, v	$[\epsilon_{xx}\ \epsilon_{yy}\ \gamma_{xy}]$	$[\tau_{xx}\ \tau_{yy}\ \tau_{xy}]$
Axisymmetric	u, v	$[\epsilon_{xx}\ \epsilon_{yy}\ \gamma_{xy}\ \epsilon_{zz}]$	$[\tau_{xx}\ \tau_{yy}\ \tau_{xy}\ \tau_{zz}]$

Linear Analysis

$$\mathbf{KU} = \mathbf{R}$$

Stiffness matrix

$$\mathbf{K} = \int_V \mathbf{B}^T\mathbf{C}\mathbf{B}\, dV$$

Linear Stress-strain matrix C (plane stress)

$$\boldsymbol{\tau} = \mathbf{C}\boldsymbol{\epsilon}$$

$$\frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

Strain-displacement matrix B (plane stress)

$$\boldsymbol{\epsilon} = \mathbf{B}\hat{\mathbf{u}}$$

?

Strains & displacements

$$\boldsymbol{\epsilon}^T = [\epsilon_{xx} \quad \epsilon_{yy} \quad \gamma_{xy}]$$

$$\epsilon_{xx} = \frac{\partial u}{\partial x}; \qquad \epsilon_{yy} = \frac{\partial v}{\partial y}; \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

Displacements

$$u = \sum_{i=1}^{q} h_i u_i; \qquad v = \sum_{i=1}^{q} h_i v_i;$$

Interpolation functions h

	$i = 5$	$i = 6$	$i = 7$	$i = 8$	$i = 9$
$h_1 = \frac{1}{4}(1+r)(1+s)$	$-\frac{1}{2}h_5$			$-\frac{1}{2}h_8$	$-\frac{1}{4}h_9$
$h_2 = \frac{1}{4}(1-r)(1+s)$	$-\frac{1}{2}h_5$	$-\frac{1}{2}h_6$			$-\frac{1}{4}h_9$
$h_3 = \frac{1}{4}(1-r)(1-s)$		$-\frac{1}{2}h_6$	$-\frac{1}{2}h_7$		$-\frac{1}{4}h_9$
$h_4 = \frac{1}{4}(1+r)(1-s)$			$-\frac{1}{2}h_7$	$-\frac{1}{2}h_8$	$-\frac{1}{4}h_9$
$h_5 = \frac{1}{2}(1-r^2)(1+s)$					$-\frac{1}{2}h_9$
$h_6 = \frac{1}{2}(1-s^2)(1-r)$					$-\frac{1}{2}h_9$
$h_7 = \frac{1}{2}(1-r^2)(1-s)$					$-\frac{1}{2}h_9$
$h_8 = \frac{1}{2}(1-s^2)(1+r)$					$-\frac{1}{2}h_9$
$h_9 = (1-r^2)(1-s^2)$					

Transformation - the Jacobian J

$$\begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial s} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} \quad \text{or} \quad \frac{\partial}{\partial \mathbf{r}} = \mathbf{J} \frac{\partial}{\partial \mathbf{x}}$$

$$\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix}_{\text{at } r=r_i,\ s=s_j} = \mathbf{J}_{ij}^{-1} \begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial s} \end{bmatrix}_{\text{at } r=r_i,\ s=s_j}$$

Strain-displacement matrix B

4 node Element

$$\begin{bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{bmatrix}_{\substack{\text{at } r=r_i \\ s=s_j}} = \frac{1}{4}\mathbf{J}_{ij}^{-1}\begin{bmatrix} 1+s_j & 0 & -(1+s_j) & 0 & -(1-s_j) & 0 & 1-s_j & 0 \\ 1+r_i & 0 & 1-r_i & 0 & -(1-r_i) & 0 & -(1+r_i) & 0 \end{bmatrix}\hat{\mathbf{u}} \qquad \text{(a)}$$

and

$$\begin{bmatrix} \dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial y} \end{bmatrix}_{\substack{\text{at } r=r_i \\ s=s_j}} = \frac{1}{4}\mathbf{J}_{ij}^{-1}\begin{bmatrix} 0 & 1+s_j & 0 & -(1+s_j) & 0 & -(1-s_j) & 0 & 1-s_j \\ 0 & 1+r_i & 0 & 1-r_i & 0 & -(1-r_i) & 0 & -(1+r_i) \end{bmatrix}\hat{\mathbf{u}} \qquad \text{(b)}$$

where $\hat{\mathbf{u}}^T = [u_1 \quad v_1 \quad u_2 \quad v_2 \quad u_3 \quad v_3 \quad u_4 \quad v_4]$

Strain-displacement matrix B

$$\left.\begin{aligned}\frac{\partial u}{\partial x} &= \sum_{i=1}^{4} \frac{\partial h_i}{\partial x} u_i \\ \frac{\partial u}{\partial y} &= \sum_{i=1}^{4} \frac{\partial h_i}{\partial y} u_i\end{aligned}\right\}$$

$$\left.\begin{aligned}\frac{\partial v}{\partial x} &= \sum_{i=1}^{4} \frac{\partial h_i}{\partial x} v_i \\ \frac{\partial v}{\partial y} &= \sum_{i=1}^{4} \frac{\partial h_i}{\partial y} v_i\end{aligned}\right\}$$

$$\boldsymbol{\epsilon} = \mathbf{B}\hat{\mathbf{u}}$$

$$\mathbf{B} = \begin{bmatrix} \frac{\partial h_1}{\partial x} & 0 & \frac{\partial h_2}{\partial x} & 0 & \frac{\partial h_3}{\partial x} & 0 & \frac{\partial h_4}{\partial x} & 0 \\ 0 & \frac{\partial h_1}{\partial y} & 0 & \frac{\partial h_2}{\partial y} & 0 & \frac{\partial h_3}{\partial y} & 0 & \frac{\partial h_4}{\partial y} \\ \frac{\partial h_1}{\partial y} & \frac{\partial h_1}{\partial x} & \frac{\partial h_2}{\partial y} & \frac{\partial h_2}{\partial x} & \frac{\partial h_3}{\partial y} & \frac{\partial h_3}{\partial x} & \frac{\partial h_4}{\partial y} & \frac{\partial h_4}{\partial x} \end{bmatrix}$$

Nonlinear, static Matrix Equations

$$ {}^{t}\mathbf{K}\mathbf{U} = {}^{t+\Delta t}\mathbf{R} - {}^{t}\mathbf{F} $$ Material nonlinear only

$$ ({}_{0}^{t}\mathbf{K}_{L} + {}_{0}^{t}\mathbf{K}_{NL})\mathbf{U} = {}^{t+\Delta t}\mathbf{R} - {}_{0}^{t}\mathbf{F} $$ Total Lagrange formulation

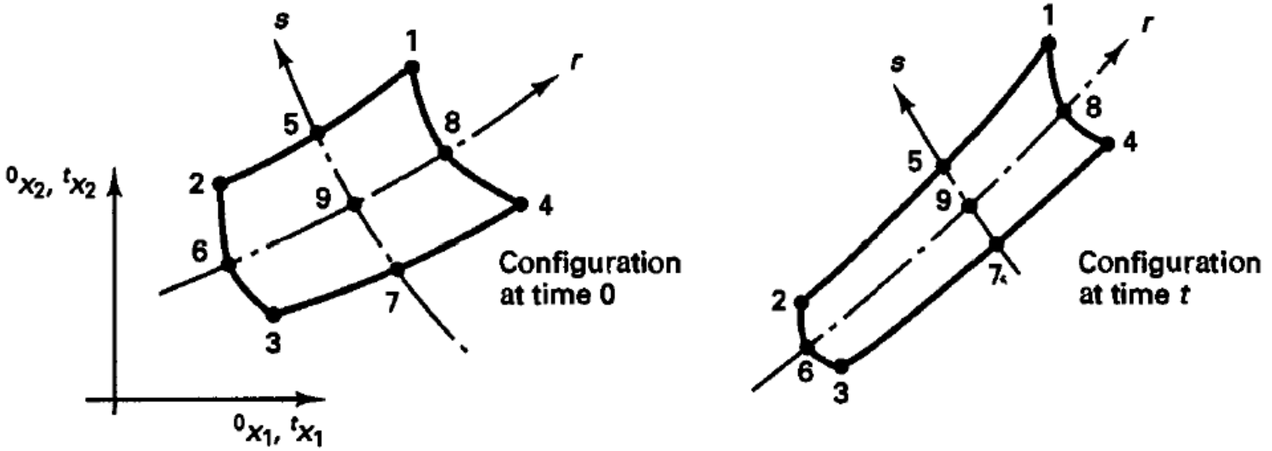
$$ ({}_{t}^{t}\mathbf{K}_{L} + {}_{t}^{t}\mathbf{K}_{NL})\mathbf{U} = {}^{t+\Delta t}\mathbf{R} - {}_{t}^{t}\mathbf{F} $$ Updated Lagrange formulation

Matrices in Total Lagrange formulation

$$ {}_{0}^{t}\mathbf{K}_{L}\hat{\mathbf{u}} = \left(\int_{{}^{0}V} {}_{0}^{t}\mathbf{B}_{L}^{T}\, {}_{0}\mathbf{C}\, {}_{0}^{t}\mathbf{B}_{L}\, d^{0}V\right)\hat{\mathbf{u}} $$ Stiffness Matrix – linear part

$$ {}_{0}^{t}\mathbf{K}_{NL}\hat{\mathbf{u}} = \left(\int_{{}^{0}V} {}_{0}^{t}\mathbf{B}_{NL}^{T}\, {}_{0}^{t}\mathbf{S}\, {}_{0}^{t}\mathbf{B}_{NL}\, d^{0}V\right)\hat{\mathbf{u}} $$ Stiffness Matrix – nonlinear part

$$ {}_{0}^{t}\mathbf{F} = \int_{{}^{0}V} {}_{0}^{t}\mathbf{B}_{L}^{T}\, {}_{0}^{t}\hat{\mathbf{S}}\, d^{0}V $$ Vector of nodal point forces



Local coordinates

$$^0x_1 = \sum_{k=1}^{N} h_k\, {}^0x_1^k; \qquad {}^0x_2 = \sum_{k=1}^{N} h_k\, {}^0x_2^k$$

$$^tx_1 = \sum_{k=1}^{N} h_k\, {}^tx_1^k; \qquad {}^tx_2 = \sum_{k=1}^{N} h_k\, {}^tx_2^k$$

Deformations

$$^tu_1 = \sum_{k=1}^{N} h_k\, {}^tu_1^k; \qquad {}^tu_2 = \sum_{k=1}^{N} h_k\, {}^tu_2^k$$

$$u_1 = \sum_{k=1}^{N} h_k\, u_1^k; \qquad u_2 = \sum_{k=1}^{N} h_k\, u_2^k$$

Required Derivatives

$$\frac{\partial^t u_i}{\partial^0 x_j} = \sum_{k=1}^{N} \left(\frac{\partial h_k}{\partial^0 x_j}\right) {}^t u_i^k$$

$$\frac{\partial u_i}{\partial^0 x_j} = \sum_{k=1}^{N} \left(\frac{\partial h_k}{\partial^0 x_j}\right) u_i^k \qquad \begin{array}{l} i = 1, 2 \\ j = 1, 2 \end{array}$$

$$\frac{\partial u_i}{\partial^t x_j} = \sum_{k=1}^{N} \left(\frac{\partial h_k}{\partial^t x_j}\right) u_i^k$$

$$\frac{\partial^t x_1}{\partial r} = \sum_{k=1}^{N} \frac{\partial h_k}{\partial r} \, {}^t x_1^k$$

Transformation – the Jacobian Matrix

Chain rule

$$\begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial s} \end{bmatrix} = {}^t\mathbf{J} \begin{bmatrix} \frac{\partial}{\partial {}^t x_1} \\ \frac{\partial}{\partial {}^t x_2} \end{bmatrix}$$

with

$${}^t\mathbf{J} = \begin{bmatrix} \frac{\partial {}^t x_1}{\partial r} & \frac{\partial {}^t x_2}{\partial r} \\ \frac{\partial {}^t x_1}{\partial s} & \frac{\partial {}^t x_2}{\partial s} \end{bmatrix}$$

Inverse Jacobian Matrix

$$\begin{bmatrix} \frac{\partial}{\partial {}^t x_1} \\ \frac{\partial}{\partial {}^t x_2} \end{bmatrix} = \frac{1}{\det {}^t\mathbf{J}} \begin{bmatrix} \frac{\partial {}^t x_2}{\partial s} & -\frac{\partial {}^t x_2}{\partial r} \\ -\frac{\partial {}^t x_1}{\partial s} & \frac{\partial {}^t x_1}{\partial r} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial s} \end{bmatrix}$$

Helpful table 6.5

2. *Linear strain-displacement transformation matrix*

Using ${}_0\mathbf{e} = {}_0^t\mathbf{B}_L \hat{\mathbf{u}}$

where ${}_0\mathbf{e}^T = [{}_0e_{11} \quad {}_0e_{22} \quad 2\,{}_0e_{12} \quad {}_0e_{33}]$; $\qquad \hat{\mathbf{u}}^T = [u_1^1 \quad u_2^1 \quad u_1^2 \quad u_2^2 \cdots u_1^N \quad u_2^N]$

and ${}_0^t\mathbf{B}_L = {}_0^t\mathbf{B}_{L0} + {}_0^t\mathbf{B}_{L1}$

$$
{}_0^t\mathbf{B}_{L0} = \begin{bmatrix}
{}_0h_{1,1} & 0 & {}_0h_{2,1} & 0 & {}_0h_{3,1} & 0 & \cdots & {}_0h_{N,1} & 0 \\
0 & {}_0h_{1,2} & 0 & {}_0h_{2,2} & 0 & {}_0h_{3,2} & \cdots & 0 & {}_0h_{N,2} \\
{}_0h_{1,2} & {}_0h_{1,1} & {}_0h_{2,2} & {}_0h_{2,1} & {}_0h_{3,2} & {}_0h_{3,1} & \cdots & {}_0h_{N,2} & {}_0h_{N,1} \\
\dfrac{h_1}{{}^0\bar{x}_1} & 0 & \dfrac{h_2}{{}^0\bar{x}_1} & 0 & \dfrac{h_3}{{}^0\bar{x}_1} & 0 & \cdots & \dfrac{h_N}{{}^0\bar{x}_1} & 0
\end{bmatrix}
$$

axisymmetric

where ${}_0h_{k,j} = \dfrac{\partial h_k}{\partial\, {}^0x_j}$; $\qquad u_j^k = {}^{t+\Delta t}u_j^k - {}^tu_j^k$; $\qquad {}^0\bar{x}_1 = \sum_{k=1}^{N} h_k\, {}^0x_1^k$; $\qquad N$ = number of nodes

and

$$
{}_0^t\mathbf{B}_{L1} = \begin{bmatrix}
l_{11}\,{}_0h_{1,1} & l_{21}\,{}_0h_{1,1} & l_{11}\,{}_0h_{2,1} & l_{21}\,{}_0h_{2,1} & \cdots & l_{11}\,{}_0h_{N,1} & l_{21}\,{}_0h_{N,1} \\
l_{12}\,{}_0h_{1,2} & l_{22}\,{}_0h_{1,2} & l_{12}\,{}_0h_{2,2} & l_{22}\,{}_0h_{2,2} & \cdots & l_{12}\,{}_0h_{N,2} & l_{22}\,{}_0h_{N,2} \\
(l_{11}\,{}_0h_{1,2} + l_{12}\,{}_0h_{1,1}) & (l_{21}\,{}_0h_{1,2} + l_{22}\,{}_0h_{1,1}) & (l_{11}\,{}_0h_{2,2} + l_{12}\,{}_0h_{2,1}) & (l_{21}\,{}_0h_{2,2} + l_{22}\,{}_0h_{2,1}) & \cdots & (l_{11}\,{}_0h_{N,2} + l_{12}\,{}_0h_{N,1}) & (l_{21}\,{}_0h_{N,2} + l_{22}\,{}_0h_{N,1}) \\
l_{33}\dfrac{h_1}{{}^0\bar{x}_1} & 0 & l_{33}\dfrac{h_2}{{}^0\bar{x}_1} & 0 & \cdots & l_{33}\dfrac{h_N}{{}^0\bar{x}_1} & 0
\end{bmatrix}
$$

axisymmetric

where $l_{11} = \sum_{k=1}^{N} {}_0h_{k,1}\, {}^tu_1^k$; $\qquad l_{22} = \sum_{k=1}^{N} {}_0h_{k,2}\, {}^tu_2^k$; $\qquad l_{21} = \sum_{k=1}^{N} {}_0h_{k,1}\, {}^tu_2^k$; $\qquad l_{12} = \sum_{k=1}^{N} {}_0h_{k,2}\, {}^tu_1^k$;

$$
l_{33} = \frac{\sum_{k=1}^{N} h_k\, {}^tu_1^k}{{}^0\bar{x}_1}
$$

Helpful table 6.5

3. *Nonlinear strain-displacement transformation matrix*

$$
{}^t_0\mathbf{B}_{NL} = \begin{bmatrix}
{}_0h_{1,1} & 0 & {}_0h_{2,1} & 0 & {}_0h_{3,1} & 0 & \cdots & {}_0h_{N,1} & 0 \\
{}_0h_{1,2} & 0 & {}_0h_{2,2} & 0 & {}_0h_{3,2} & 0 & \cdots & {}_0h_{N,2} & 0 \\
0 & {}_0h_{1,1} & 0 & {}_0h_{2,1} & 0 & {}_0h_{3,1} & \cdots & 0 & {}_0h_{N,1} \\
0 & {}_0h_{1,2} & 0 & {}_0h_{2,2} & 0 & {}_0h_{3,2} & \cdots & 0 & {}_0h_{N,2} \\
\dfrac{h_1}{{}^0\bar{x}_1} & 0 & \dfrac{h_2}{{}^0\bar{x}_1} & 0 & \dfrac{h_3}{{}^0\bar{x}_1} & 0 & \cdots & \dfrac{h_N}{{}^0\bar{x}_1} & 0
\end{bmatrix}
$$

axisymmetric (last row)

Extension to 3D

$$^{0}x_i = \sum_{k=1}^{N} h_k \, {}^{0}x_i^k; \qquad {}^{t}x_i = \sum_{k=1}^{N} h_k \, {}^{t}x_i^k; \qquad i = 1, 2, 3$$

$$^{t}u_i = \sum_{k=1}^{N} h_k \, {}^{t}u_i^k; \qquad u_i = \sum_{k=1}^{N} h_k \, u_i^k; \qquad i = 1, 2, 3$$

Table 6.6 ...

Example 6.18 p. 554