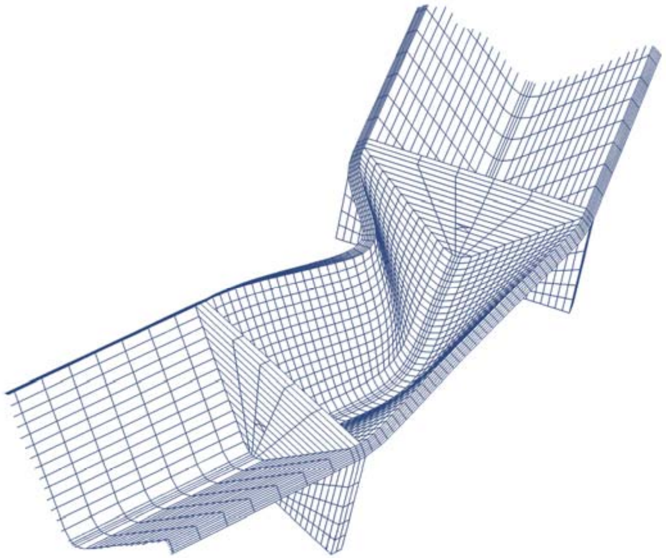
The Finite Element Method for the Analysis of Non-Linear and Dynamic Systems



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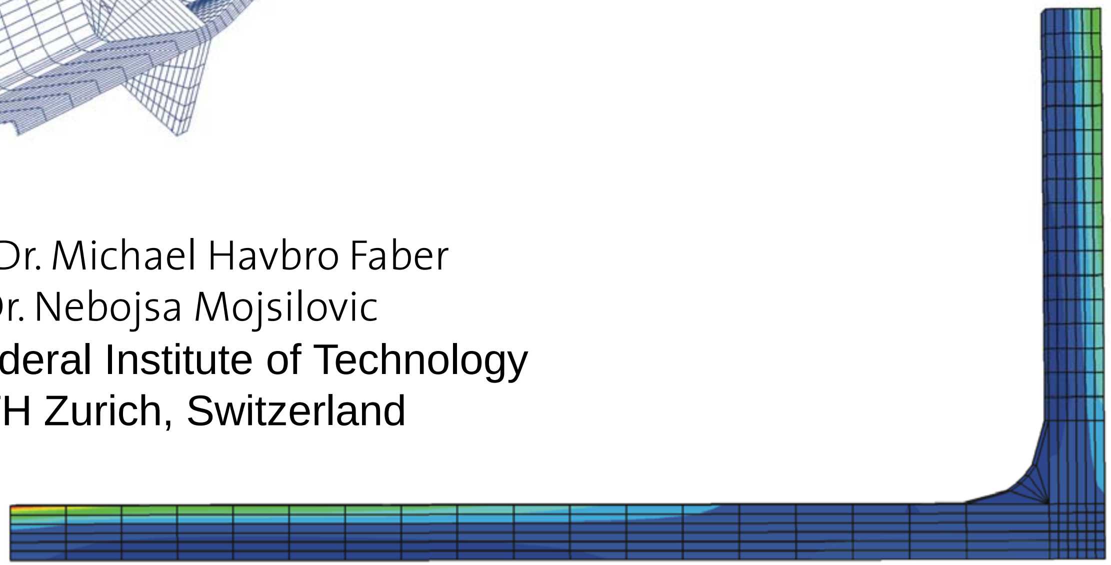
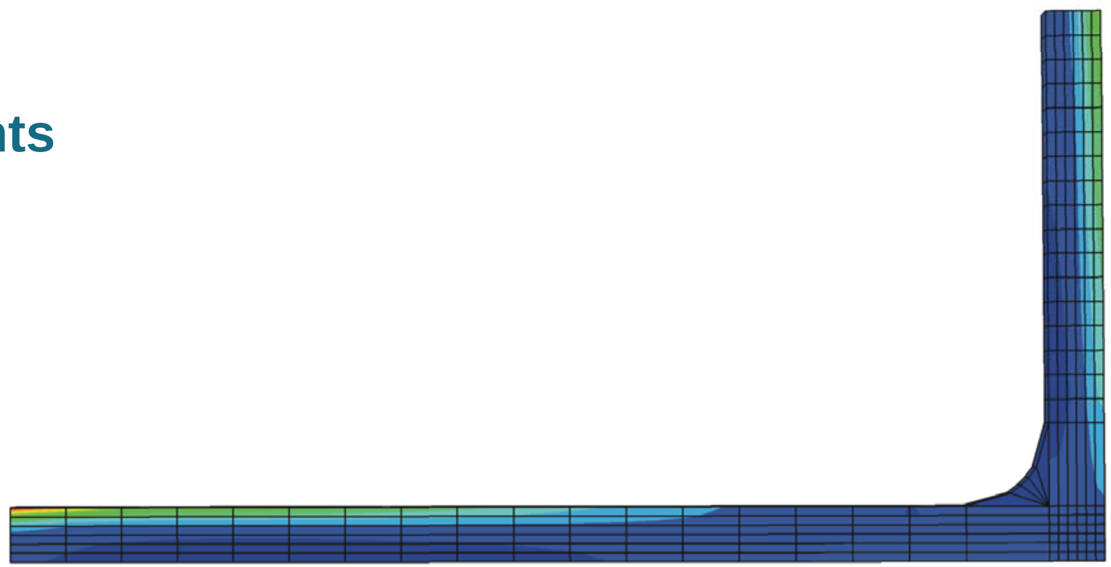


Contents of Today's Lecture

- Aim of the lecture
- Displacement-based isoparametric continuum finite elements
- General matrix equations of displacement-based continuum elements
- Truss and cable elements
- Example



Displacement-based isoparametric continuum finite elements

- **Linearisation of the principle of virtual work with respect to finite element variables**

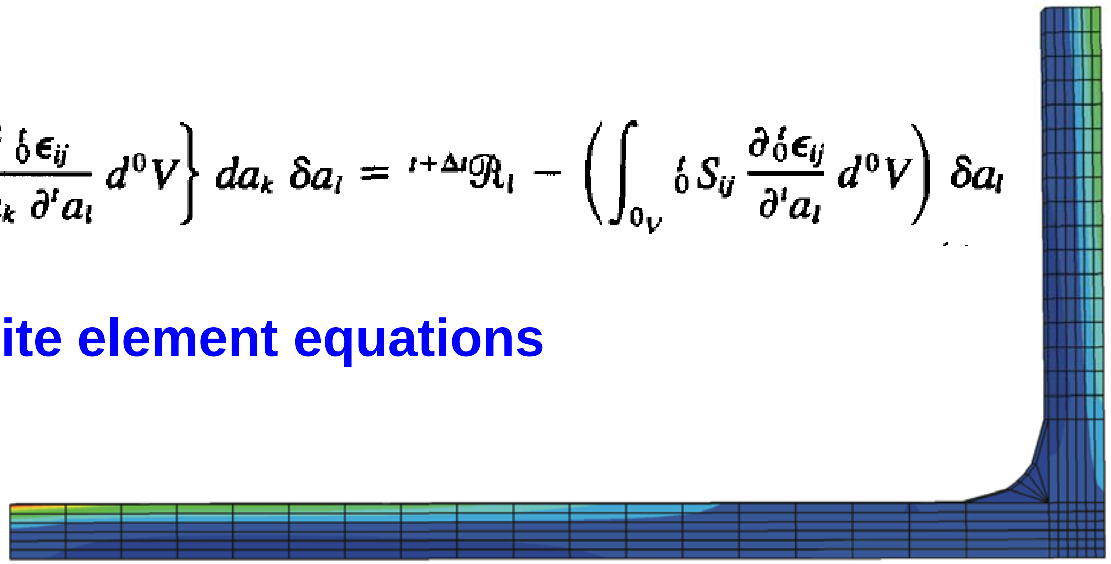
TL formulation

$$\int_{{}^0V} {}^{t+\Delta t}_{\;\;0}S_{ij}\, \delta\, {}^{t+\Delta t}_{\;\;0}\epsilon_{ij}\, d^0V = {}^{t+\Delta t}\mathcal{R}$$

At the end we obtain

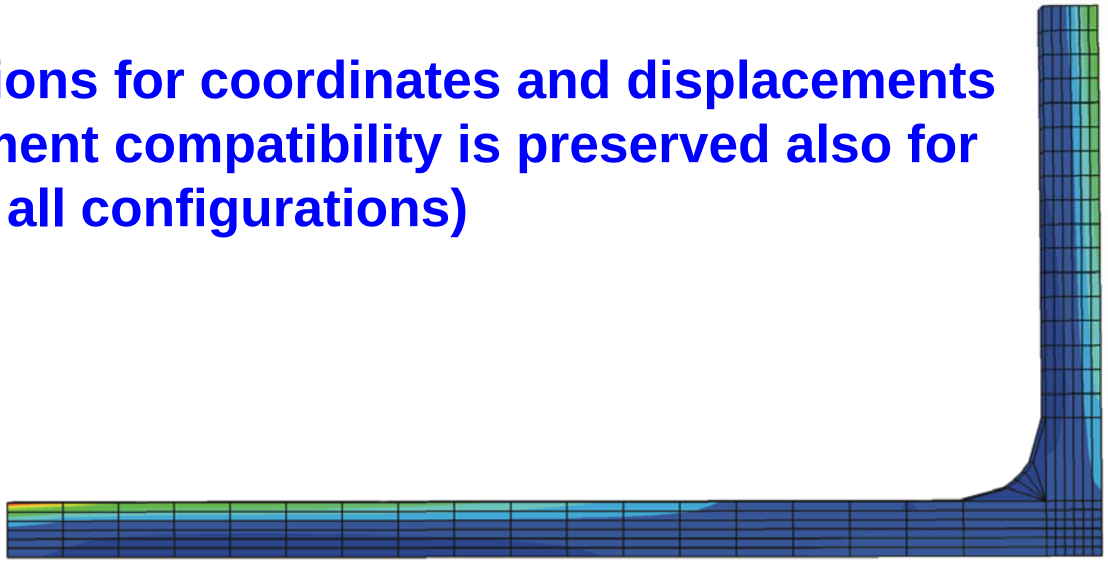
$$\left\{\int_{{}^0V} {}_0C_{ijrs} \frac{\partial\, {}^t_0\epsilon_{rs}}{\partial\, {}^t a_k}\frac{\partial\, {}^t_0\epsilon_{ij}}{\partial\, {}^t a_l}\, d^0V + \int_{{}^0V} {}^t_0S_{ij} \frac{\partial^2\, {}^t_0\epsilon_{ij}}{\partial\, {}^t a_k\, \partial\, {}^t a_l}\, d^0V\right\} da_k\, \delta a_l = {}^{t+\Delta t}\mathcal{R}_l - \left(\int_{{}^0V} {}^t_0S_{ij}\frac{\partial\, {}^t_0\epsilon_{ij}}{\partial\, {}^t a_l}\, d^0V\right)\delta a_l$$

and can now derive the finite element equations



General matrix equations of displacement-based continuum elements

- The selection of interpolation functions and their implementation (displacements and coordinates)
- By invoking the linearized principle of virtual displacements for each nodal point displacement the governing equations are obtained
- Same interpolation functions for coordinates and displacements assure that the displacement compatibility is preserved also for incremental analysis (for all configurations)



General matrix equations of displacement-based continuum elements

- **Materially-nonlinear-only analysis:**

- Static analysis

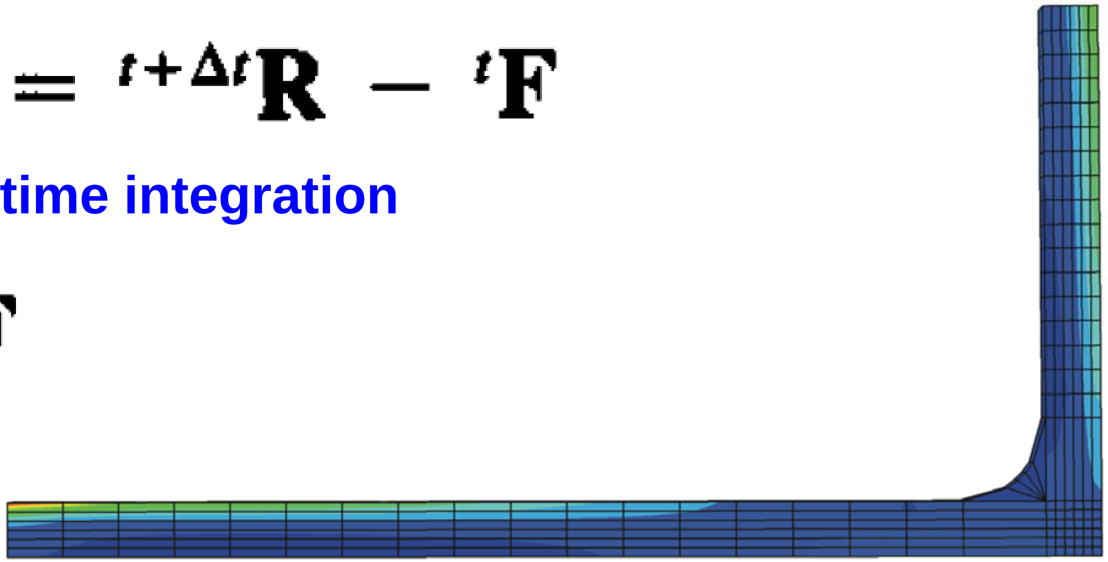
$$ {}^{t}\mathbf{K}\mathbf{U} = {}^{t+\Delta t}\mathbf{R} - {}^{t}\mathbf{F} $$

- Dynamic analysis, implicit time integration

$$ \mathbf{M}\,{}^{t+\Delta t}\ddot{\mathbf{U}} + {}^{t}\mathbf{K}\mathbf{U} = {}^{t+\Delta t}\mathbf{R} - {}^{t}\mathbf{F} $$

- Dynamic analysis, explicit time integration

$$ \mathbf{M}\,{}^{t}\ddot{\mathbf{U}} = {}^{t}\mathbf{R} - {}^{t}\mathbf{F} $$



General matrix equations of displacement-based continuum elements

- **TL (Total Lagrange) formulaton:**

- Static analysis

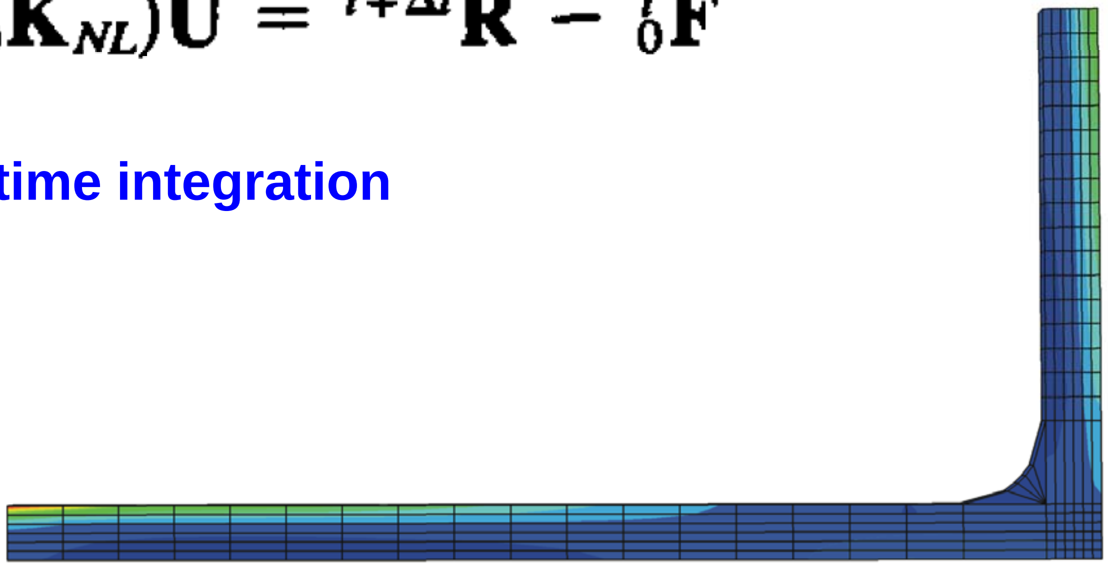
$$({}^{t}_{0}\mathbf{K}_{L} + {}^{t}_{0}\mathbf{K}_{NL})\mathbf{U} = {}^{t+\Delta t}\mathbf{R} - {}^{t}_{0}\mathbf{F}$$

- Dynamic analysis, implicit time integration

$$\mathbf{M}\,{}^{t+\Delta t}\ddot{\mathbf{U}} + ({}^{t}_{0}\mathbf{K}_{L} + {}^{t}_{0}\mathbf{K}_{NL})\mathbf{U} = {}^{t+\Delta t}\mathbf{R} - {}^{t}_{0}\mathbf{F}$$

- Dynamic analysis, explicit time integration

$$\mathbf{M}\,{}^{t}\ddot{\mathbf{U}} = {}^{t}\mathbf{R} - {}^{t}_{0}\mathbf{F}$$



General matrix equations of displacement-based continuum elements

- **UL (Updated Lagrange) formulaton:**

- Static analysis

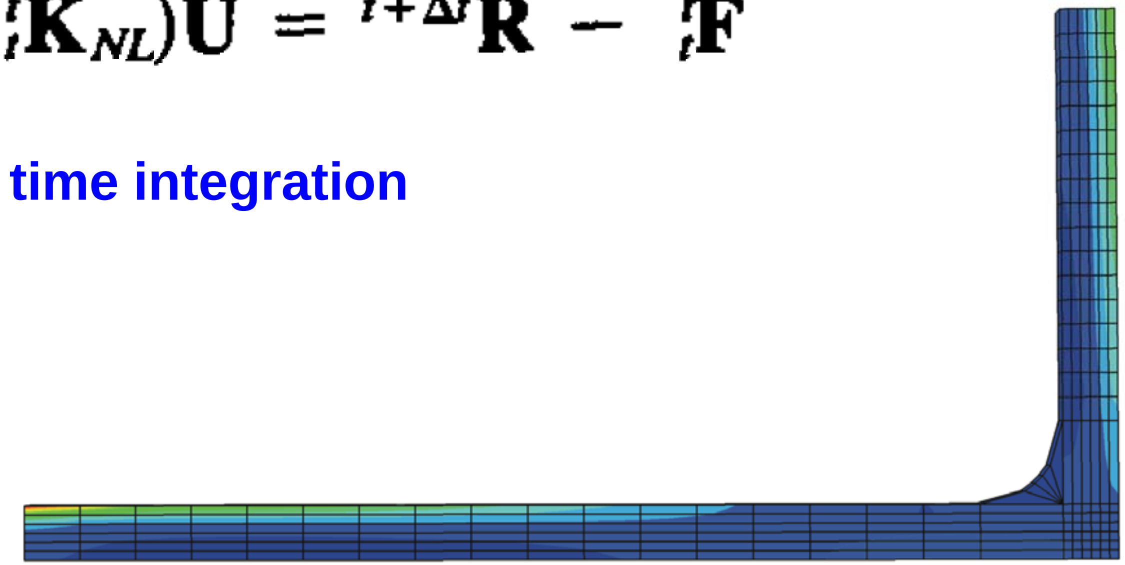
$$({}^{t}_{t}\mathbf{K}_{L} + {}^{t}_{t}\mathbf{K}_{NL})\mathbf{U} = {}^{t+\Delta t}\mathbf{R} - {}^{t}_{t}\mathbf{F}$$

- Dynamic analysis, implicit time integration

$$\mathbf{M}\,{}^{t+\Delta t}\ddot{\mathbf{U}} + ({}^{t}_{t}\mathbf{K}_{L} + {}^{t}_{t}\mathbf{K}_{NL})\mathbf{U} = {}^{t+\Delta t}\mathbf{R} - {}^{t}_{t}\mathbf{F}$$

- Dynamic analysis, explicit time integration

$$\mathbf{M}\,{}^{t}\ddot{\mathbf{U}} = {}^{t}\mathbf{R} - {}^{t}_{t}\mathbf{F}$$



General matrix equations of displacement-based continuum elements

where $\mathbf{M}$ = time-independent mass matrix

${}^t\mathbf{K}$ = linear strain incremental stiffness matrix, not including the initial displacement effect

${}^t_0\mathbf{K}_L$, ${}^t_t\mathbf{K}_L$ = linear strain incremental stiffness matrices

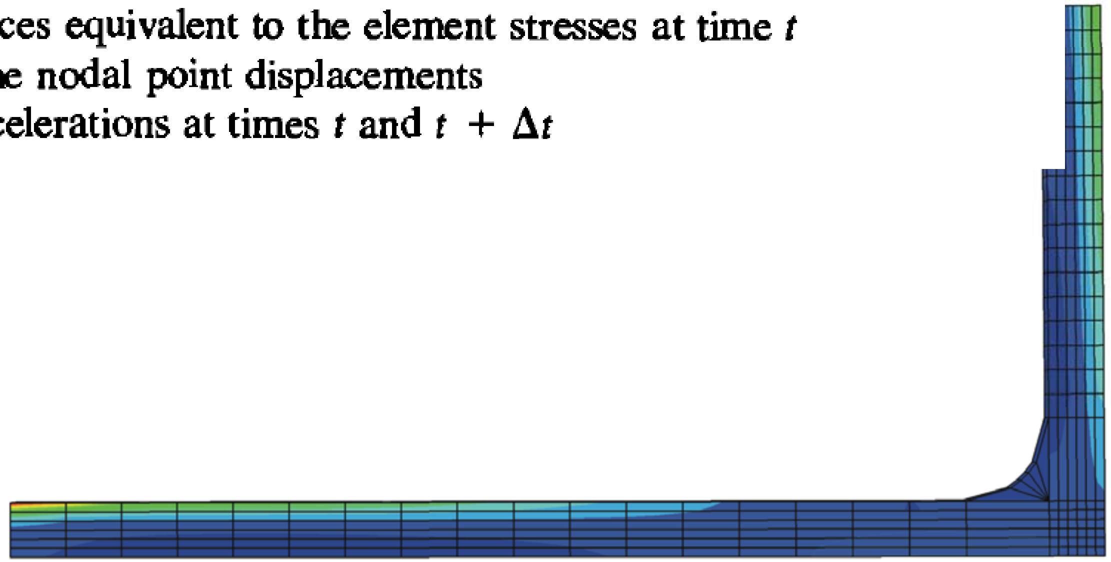
${}^t_0\mathbf{K}_{NL}$, ${}^t_t\mathbf{K}_{NL}$ = nonlinear strain (geometric or initial stress) incremental stiffness matrices

${}^{t+\Delta t}\mathbf{R}$ = vector of externally applied nodal point loads at time $t + \Delta t$; this vector is also used at time t in explicit time integration

${}^t\mathbf{F}$, ${}^t_0\mathbf{F}$, ${}^t_t\mathbf{F}$ = vectors of nodal point forces equivalent to the element stresses at time t

$\mathbf{U}$ = vector of increments in the nodal point displacements

${}^t\ddot{\mathbf{U}}$, ${}^{t+\Delta t}\ddot{\mathbf{U}}$ = vectors of nodal point accelerations at times t and $t + \Delta t$



General matrix equations of displacement-based continuum elements

$\mathbf{H}^S, \mathbf{H}$ = surface- and volume-displacement interpolation matrices

${}^{t+\Delta t}_{0}\mathbf{f}^S, {}^{t+\Delta t}_{0}\mathbf{f}^B$ = vectors of surface and body forces defined per unit area and per unit volume of the element at time 0

$\mathbf{B}_L, {}^t_0\mathbf{B}_L, {}^t_t\mathbf{B}_L$ = linear strain-displacement transformation matrices; $\mathbf{B}_L$ is equal to ${}^t_0\mathbf{B}_L$ when the initial displacement effect is neglected

${}^t_0\mathbf{B}_{NL}, {}^t_t\mathbf{B}_{NL}$ = nonlinear strain-displacement transformation matrices

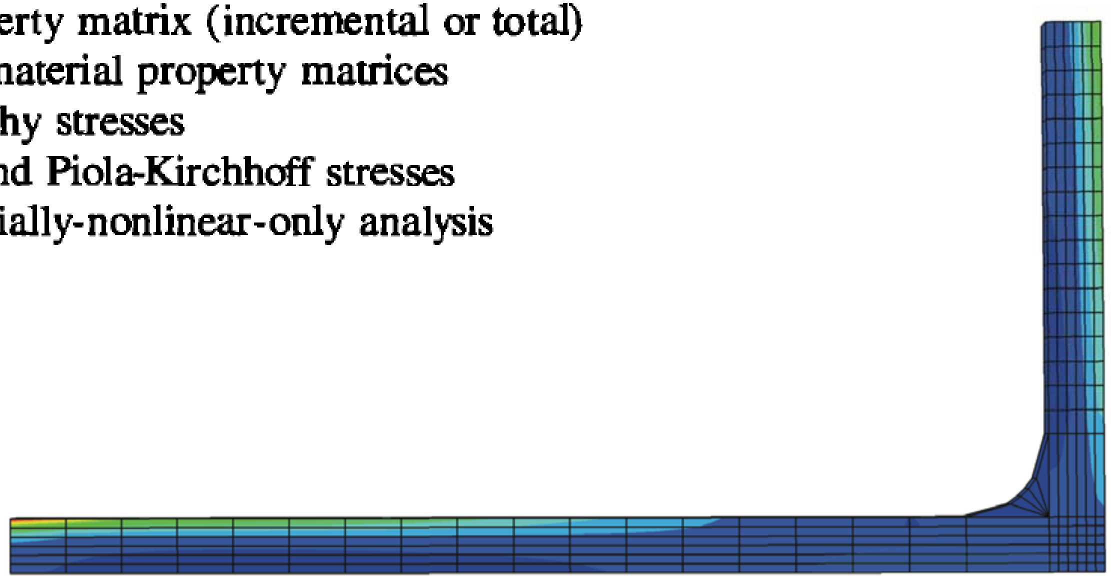
$\mathbf{C}$ = stress-strain material property matrix (incremental or total)

${}_0\mathbf{C}, {}_t\mathbf{C}$ = incremental stress-strain material property matrices

${}^t\boldsymbol{\tau}, {}^t\hat{\boldsymbol{\tau}}$ = matrix and vector of Cauchy stresses

${}^t_0\mathbf{S}, {}^t_0\hat{\mathbf{S}}$ = matrix and vector of second Piola-Kirchhoff stresses

${}^t\hat{\boldsymbol{\Sigma}}$ = vector of stresses in materially-nonlinear-only analysis



Finite element matrices (Table 6.4)

- **All analysis:**

- Integral

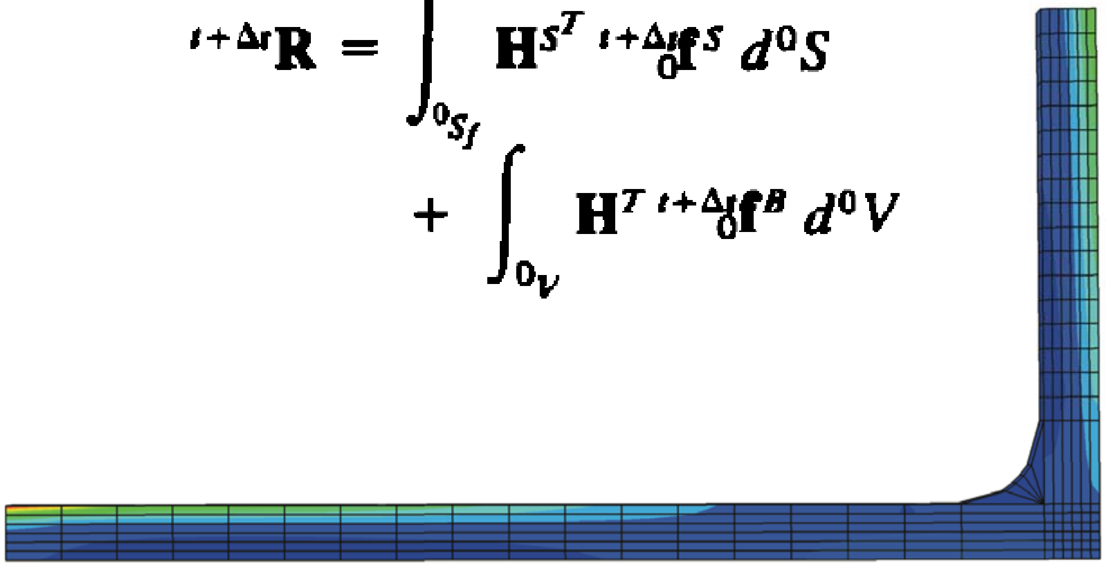
$$\int_{{}^0V} {}^0\rho \; {}^{t+\Delta t}\ddot{u}_i \, \delta u_i \, d\,{}^0V$$

$$\delta\,{}^{t+\Delta t}\mathcal{R} = \int_{{}^0S_f} {}^{t+\Delta t}_{\;\;0}f_i^S \, \delta u_i^S \, d\,{}^0S + \int_{{}^0V} {}^{t+\Delta t}_{\;\;0}f_i^B \, \delta u_i \, d\,{}^0V$$

Matrix evaluation

$$\mathbf{M}\; {}^{t+\Delta t}\ddot{\mathbf{U}} = \left(\int_{{}^0V} {}^0\rho \, \mathbf{H}^T\mathbf{H} \, d\,{}^0V \right) {}^{t+\Delta t}\ddot{\mathbf{U}}$$

$$ {}^{t+\Delta t}\mathbf{R} = \int_{{}^0S_f} \mathbf{H}^{S^T} \, {}^{t+\Delta t}_{\;\;0}\mathbf{f}^S \, d\,{}^0S + \int_{{}^0V} \mathbf{H}^T \, {}^{t+\Delta t}_{\;\;0}\mathbf{f}^B \, d\,{}^0V$$



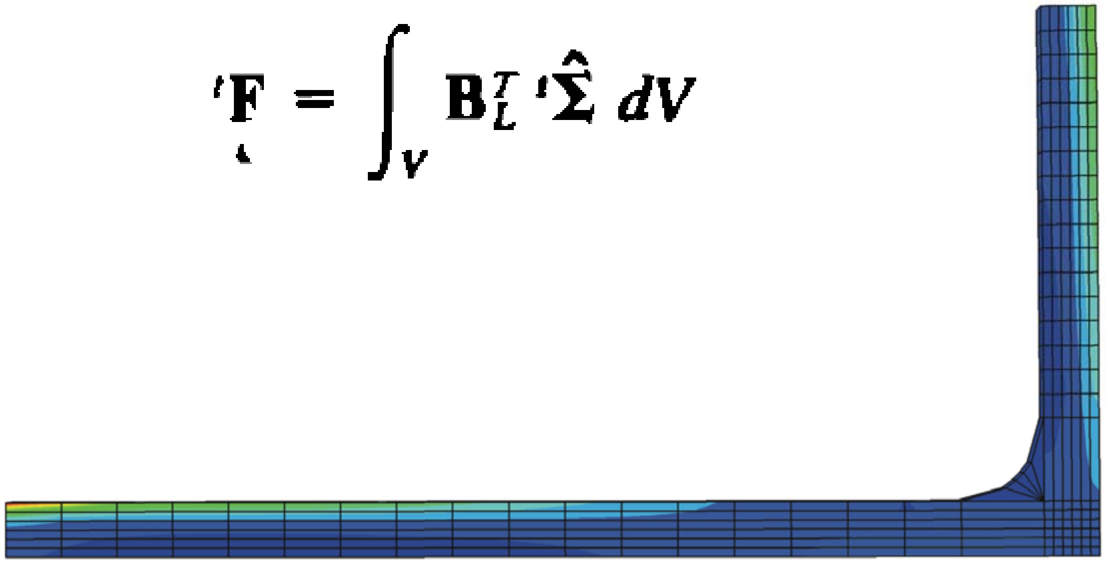
Finite element matrices (Table 6.4)

- **Materiallly-nonlinear-only:**

- Integral | **Matrix evaluation**

$$\int_V C_{ijrs}\, e_{rs}\, \delta e_{ij}\, dV \qquad {}^t\mathbf{K}\hat{\mathbf{u}} = \left(\int_V \mathbf{B}_L^T \mathbf{C} \mathbf{B}_L\, dV\right)\hat{\mathbf{u}}$$

$$\int_V {}^t\sigma_{ij}\, \delta e_{ij}\, dV \qquad {}^t\mathbf{F} = \int_V \mathbf{B}_L^T\, {}^t\hat{\boldsymbol{\Sigma}}\, dV$$



Finite element matrices (Table 6.4)

- **TL formulation:**

- Integral

$$\int_{{}^0V} {}_0C_{ijrs}\ {}_0e_{rs}\ \delta_0 e_{ij}\ d^0V$$

$$\int_{{}^0V} {}^t_0S_{ij}\ \delta_0\eta_{ij}\ d^0V$$

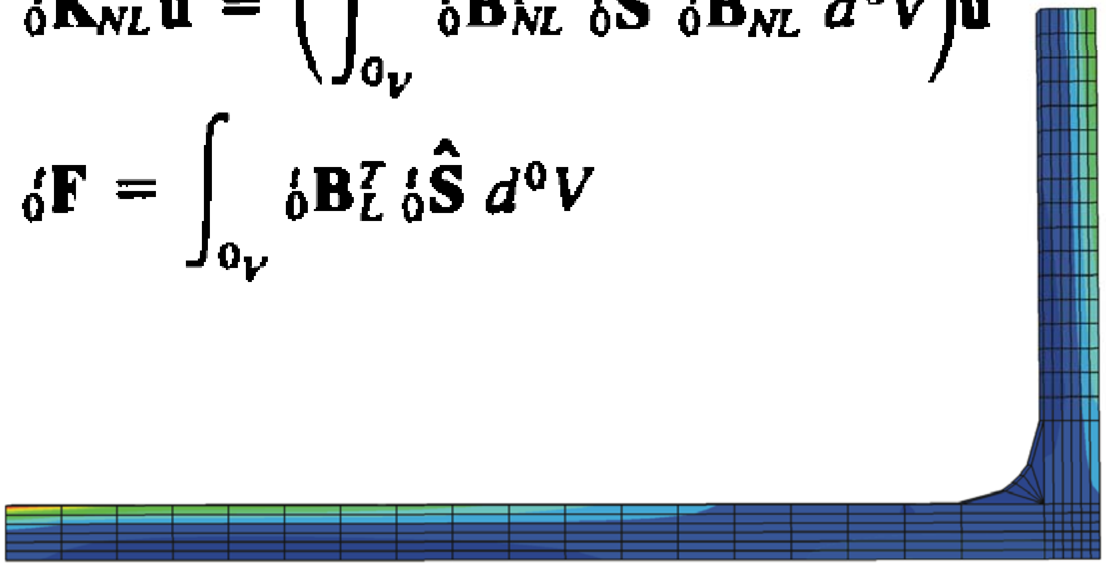
$$\int_{{}^0V} {}^t_0S_{ij}\ \delta_0 e_{ij}\ d^0V$$

Matrix evaluation

$$ {}^t_0\mathbf{K}_L\hat{\mathbf{u}} = \left(\int_{{}^0V} {}^t_0\mathbf{B}_L^T\ {}_0\mathbf{C}\ {}^t_0\mathbf{B}_L\ d^0V\right)\hat{\mathbf{u}}$$

$$ {}^t_0\mathbf{K}_{NL}\hat{\mathbf{u}} = \left(\int_{{}^0V} {}^t_0\mathbf{B}_{NL}^T\ {}^t_0\mathbf{S}\ {}^t_0\mathbf{B}_{NL}\ d^0V\right)\hat{\mathbf{u}}$$

$$ {}^t_0\mathbf{F} = \int_{{}^0V} {}^t_0\mathbf{B}_L^T\ {}^t_0\hat{\mathbf{S}}\ d^0V$$



Finite element matrices (Table 6.4)

- **UL formulation:**

- Integral

$$\int_{^tV} {}_tC_{ijrs}\, {}_te_{rs}\, \delta_t e_{ij}\, d^tV$$

$$\int_{^tV} {}^t\tau_{ij}\, \delta_t \eta_{ij}\, d^tV$$

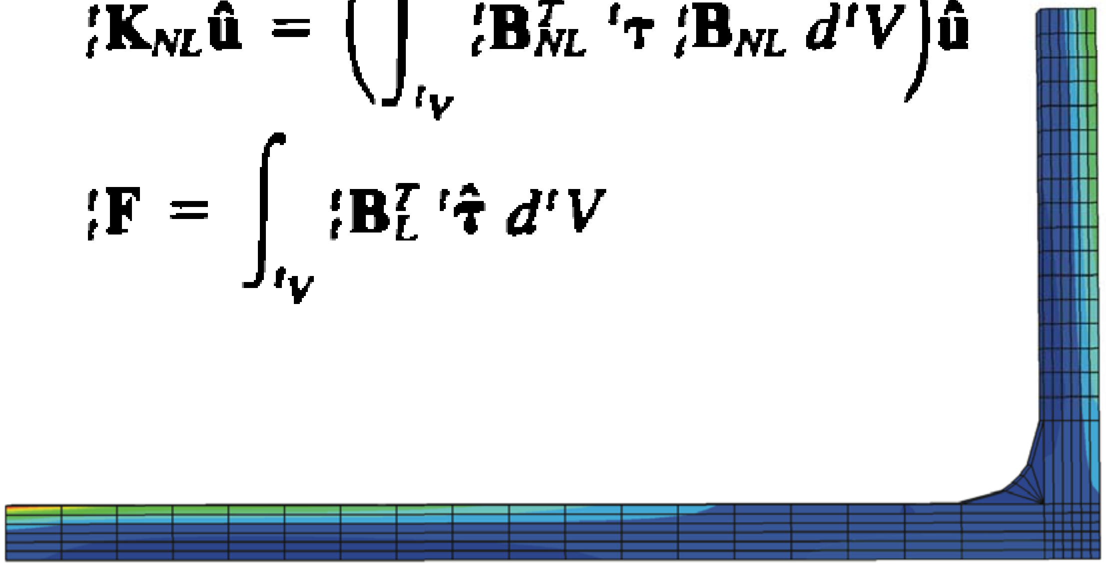
$$\int_{^tV} {}^t\tau_{ij}\, \delta_t e_{ij}\, d^tV$$

Matrix evaluation

$${}^t_t\mathbf{K}_L\hat{\mathbf{u}} \left(\int_{^tV} {}^t_t\mathbf{B}_L^T\, {}_t\mathbf{C}\, {}^t_t\mathbf{B}_L\, d^tV\right)\hat{\mathbf{u}}$$

$${}^t_t\mathbf{K}_{NL}\hat{\mathbf{u}} = \left(\int_{^tV} {}^t_t\mathbf{B}_{NL}^T\, {}^t\boldsymbol{\tau}\, {}^t_t\mathbf{B}_{NL}\, d^tV\right)\hat{\mathbf{u}}$$

$${}^t_t\mathbf{F} = \int_{^tV} {}^t_t\mathbf{B}_L^T\, {}^t\hat{\boldsymbol{\tau}}\, d^tV$$



Truss and cable elements