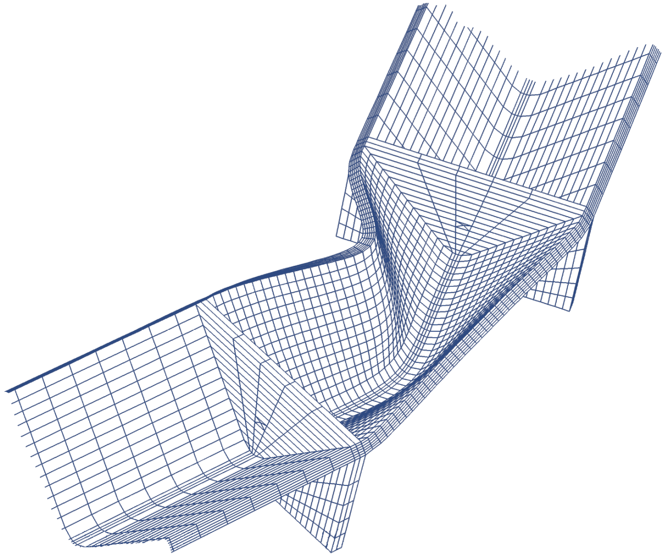
The Finite Element Method
for the Analysis of Non-Linear and Dynamic Systems



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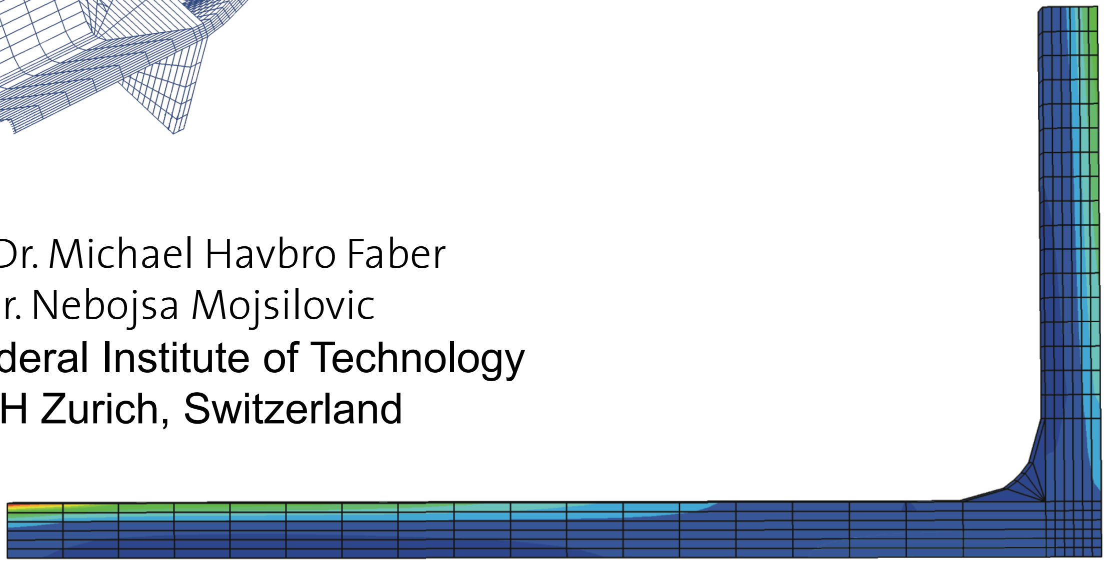
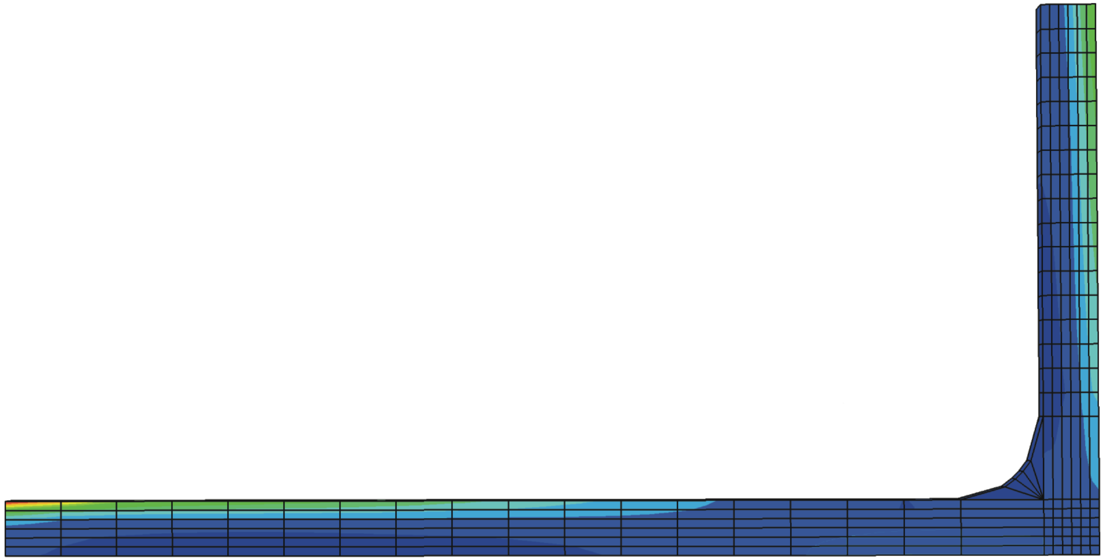


Contents of Today's Lecture

- **Motivation, overview and organization of the course**
- **Introduction to non-linear analysis**
- **Formulation of the continuum mechanics incremental equations of motion**



Motivation, overview and organization of the course

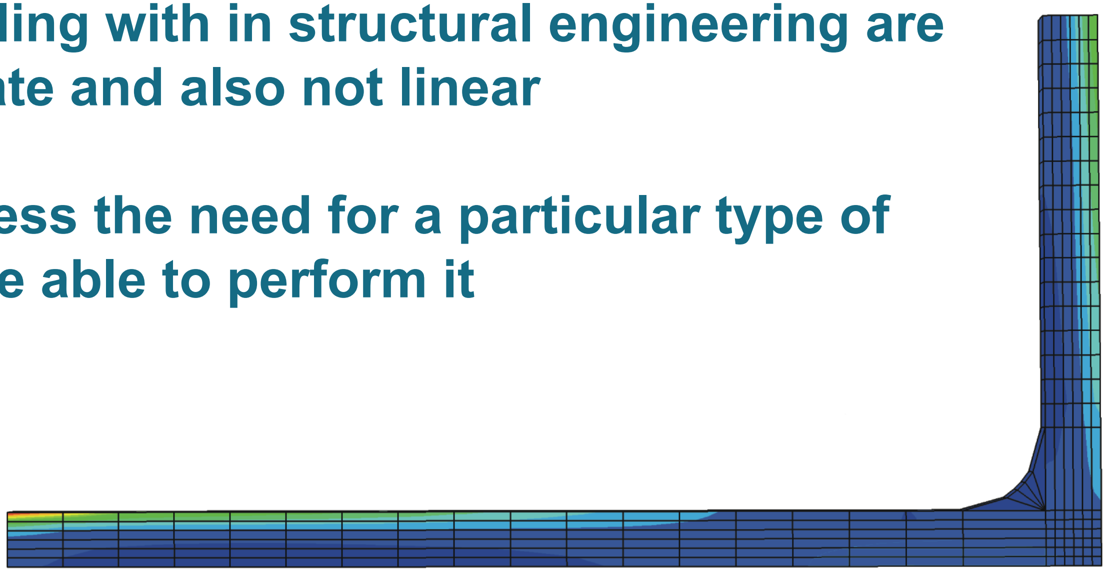
- **Motivation**

In FEM 1 we learned about the steady state analysis of linear systems

however;

the systems we are dealing with in structural engineering are generally not steady state and also not linear

We must be able to assess the need for a particular type of analysis and we must be able to perform it



Motivation, overview and organization of the course

- **Motivation**

What kind of problems are not steady state and linear?

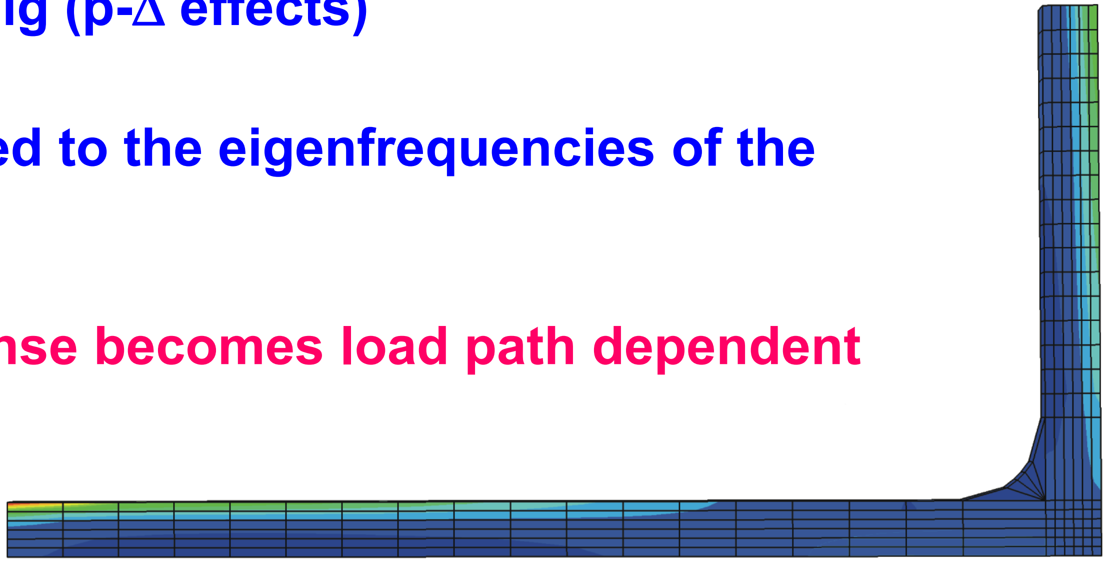
E.g. when the:

material behaves non-linearly

deformations become big (p-Δ effects)

loads vary fast compared to the eigenfrequencies of the structure

General feature: Response becomes load path dependent



Motivation, overview and organization of the course

- **Motivation**

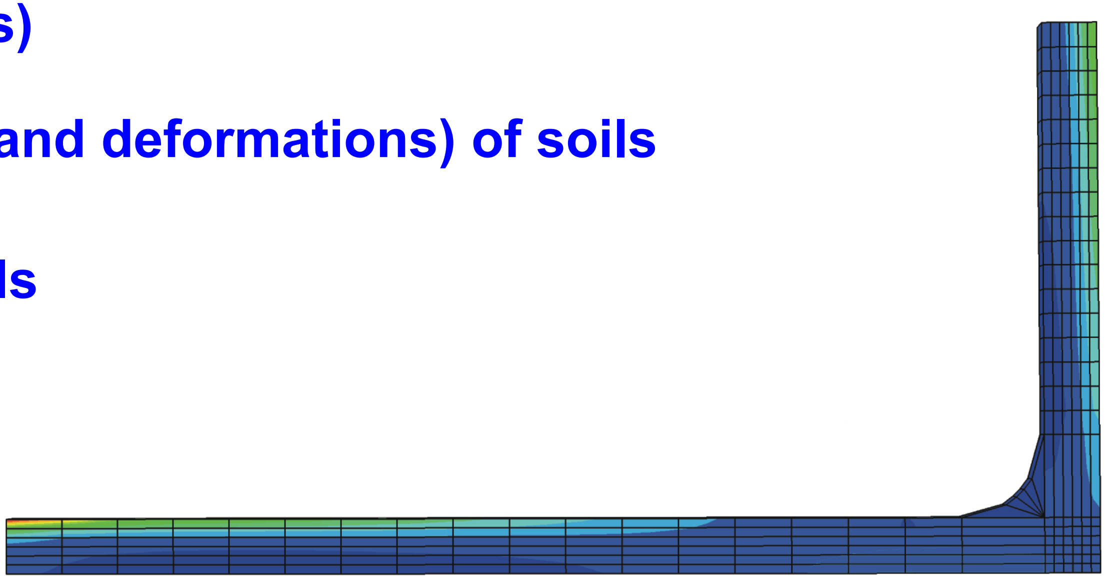
What is the „added value“ of being able to assess the non-linear non-steady state response of structures ?

E.g. assessing the;

- structural response of structures to extreme events (rock-fall, earthquake, hurricanes)

- performance (failures and deformations) of soils

- verifying simple models



Motivation, overview and organization of the course

Steady state problems (Linear/Non-linear):

The response of the system does not change over time

$$\mathbf{KU} = \mathbf{R}$$

Propagation problems (Linear/Non-linear):

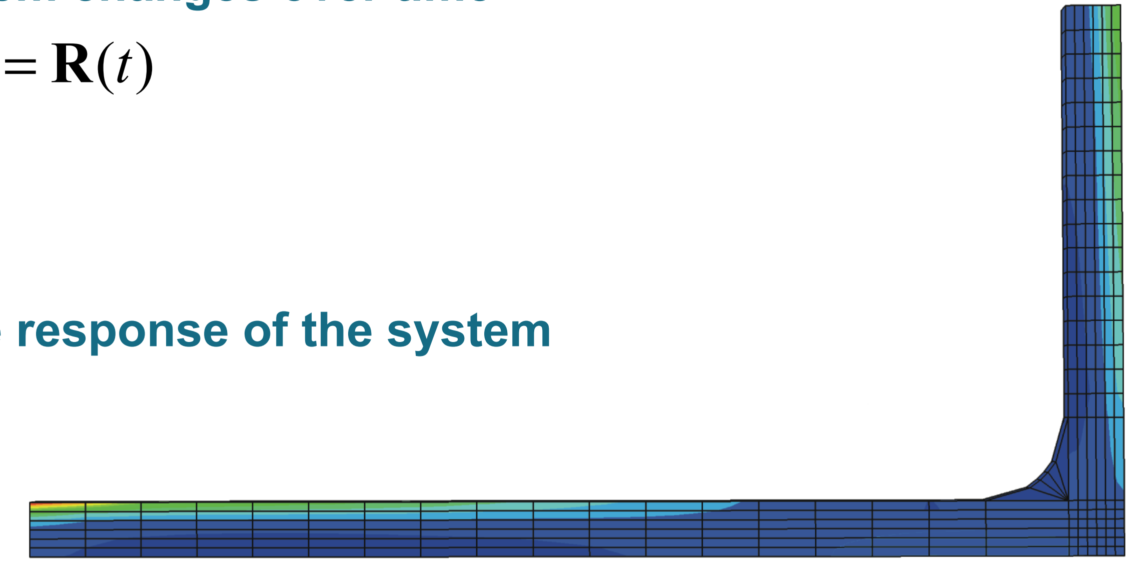
The response of the system changes over time

$$\mathbf{M}\ddot{\mathbf{U}}(t) + \mathbf{C}\dot{\mathbf{U}}(t) + \mathbf{KU}(t) = \mathbf{R}(t)$$

Eigenvalue problems:

No unique solution to the response of the system

$$\mathbf{Av} = \lambda \mathbf{Bv}$$



Motivation, overview and organization of the course

- **Organisation**

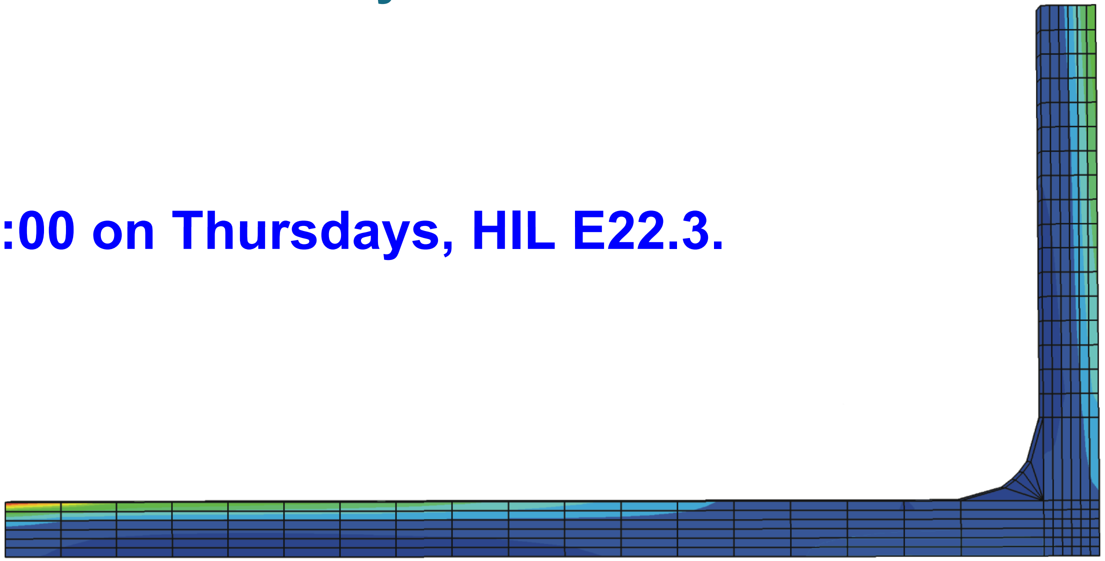
  The lectures will be given by:

  M. H. Faber and N. Mojsilovic

  Exercises will be organized/attended by:

  K. Nishijima

  Office hours: 14:00 – 16:00 on Thursdays, HIL E22.3.



Motivation, overview and organization of the course

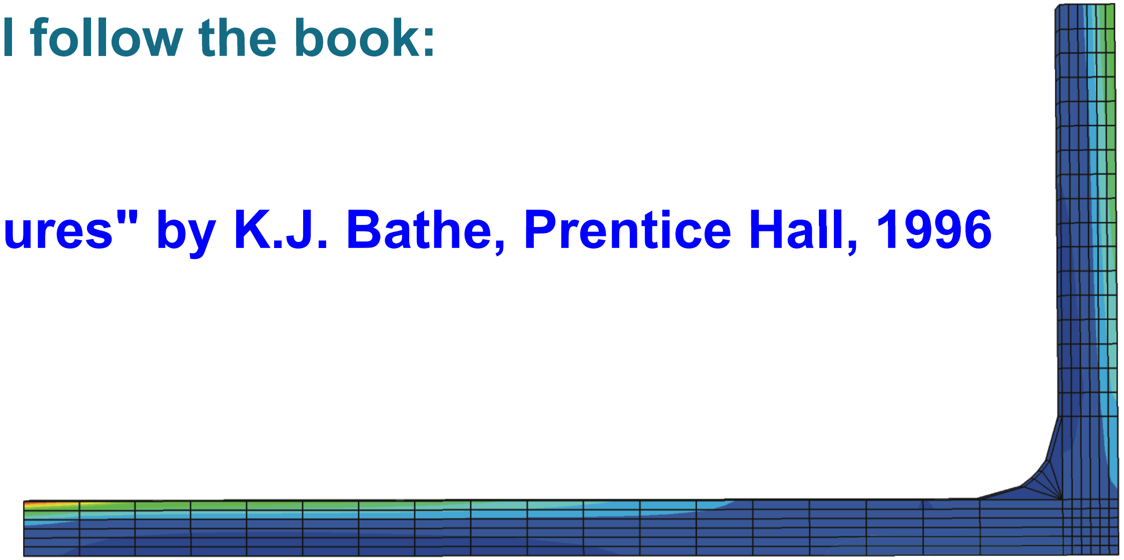
- Organisation

  PowerPoint files with the presentations will be up-loaded on our home-page one day in advance of the lectures

  http://www.ibk.ethz.ch/fa/education/FE_II

  The lecture as such will follow the book:

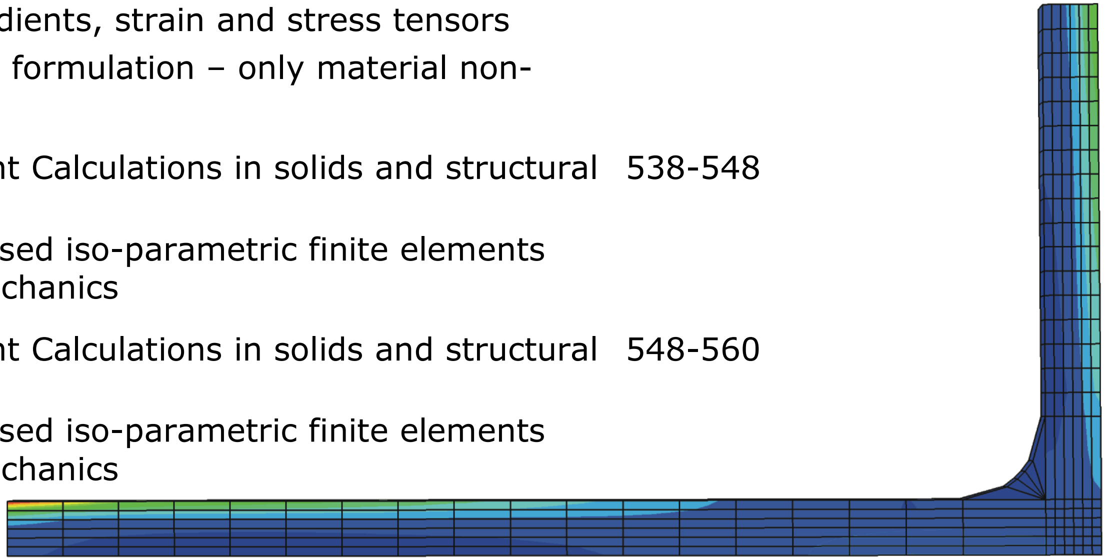
  "Finite Element Procedures" by K.J. Bathe, Prentice Hall, 1996



Motivation, overview and organization of the course

Overview

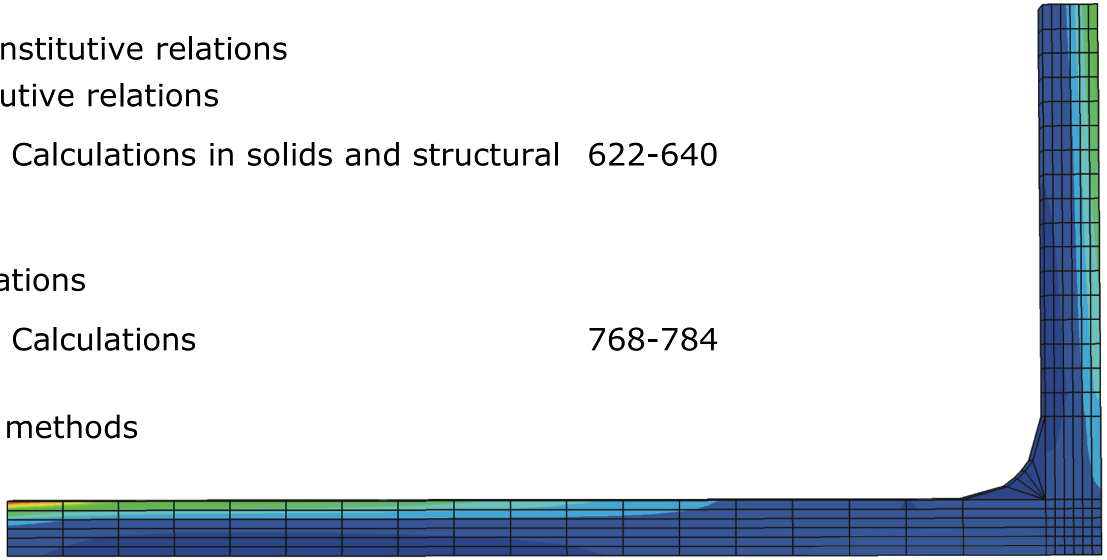
Date	Subject(s)	Course book Pages:
28.09.2007	Non-linear Finite Element Calculations in solids and structural mechanics • Introduction to non-linear calculations • The incremental approach to continuum mechanics	485-502
05.10.2007	Non-linear Finite Element Calculations in solids and structural mechanics • Deformation gradients, strain and stress tensors • The Langrangian formulation – only material non-linearity	502-528
12.10.2007	Non-linear Finite Element Calculations in solids and structural mechanics • Displacement based iso-parametric finite elements in continuum mechanics	538-548
19.10.2007	Non-linear Finite Element Calculations in solids and structural mechanics • Displacement based iso-parametric finite elements in continuum mechanics	548-560



Motivation, overview and organization of the course

Overview

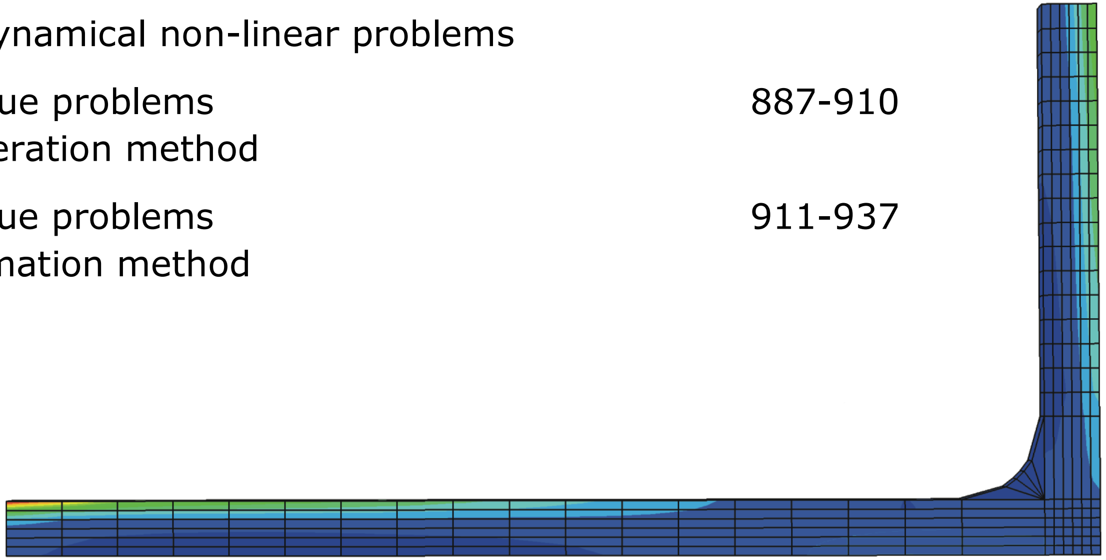
Date	Subject(s)	Course book Pages:
26.10.2007	Non-linear Finite Element Calculations in solids and structural mechanics • Total Langrangian formulation • Extended Langrangian formulation • Structural elements	561-578
02.11.2007	Non-linear Finite Element Calculations in solids and structural mechanics • Introduction of constitutive relations • Non-linear constitutive relations	581-617
09.11.2007	Non-linear Finite Element Calculations in solids and structural mechanics • Contact problems • Practical considerations	622-640
16.11.2007	Dynamical Finite Element Calculations • Introduction • Direct integration methods	768-784



Motivation, overview and organization of the course

Overview

Date	Subject(s)	Course book Pages:
23.11.2007	Dynamical Finite Element Calculations • Mode superposition	785-800
30.11.2007	Dynamical Finite Element Calculations • Analysis of direct integration methods	801-815
07.12.2007	Dynamical Finite Element Calculations • Solution of dynamical non-linear problems	824-830
14.12.2007	Solution of Eigen value problems • The vector iteration method	887-910
21.12.2007	Solution of Eigen value problems • The transformation method	911-937



Introduction to non-linear analysis

- Previsously we considered the solution of the following linear and static problem:

$$\mathbf{KU} = \mathbf{R}$$

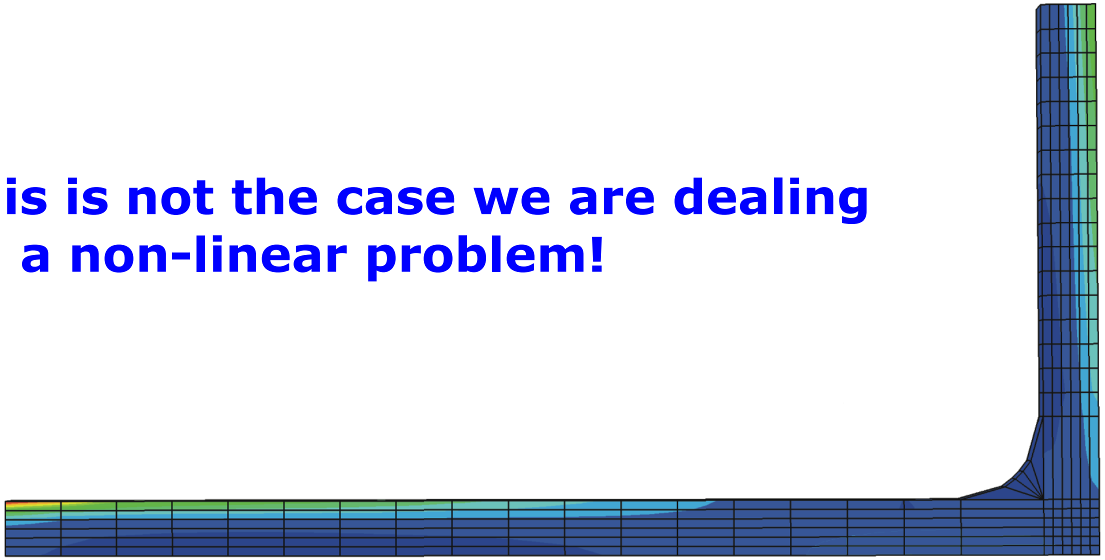
for these problems we have the convenient property of linearity, i.e:

$$\mathbf{KU}^* = \lambda\mathbf{R}$$

$$\Downarrow$$

$$\mathbf{U}^* = \lambda\mathbf{U}$$

If this is not the case we are dealing with a non-linear problem!



Introduction to non-linear analysis

- Previsously we considered the solution of the following linear and static problem:

$$\mathbf{KU} = \mathbf{R}$$

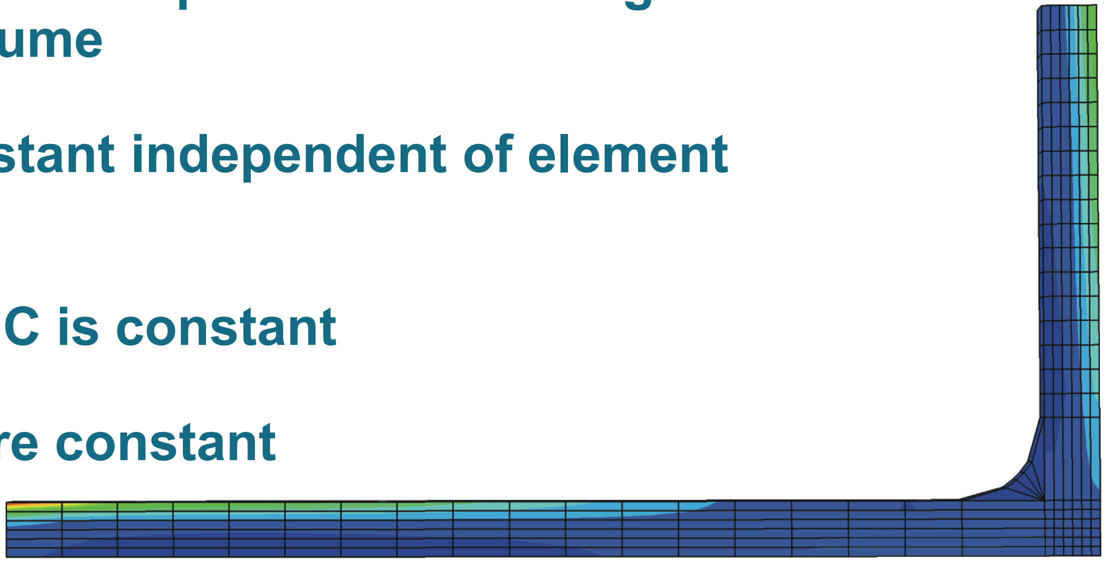
we assumed:

small displacements when developing the stiffness matrix K and the load vector R, because we performed all integrations over the original element volume

that the B matrix is constant independent of element displacements

the stress-strain matrix C is constant

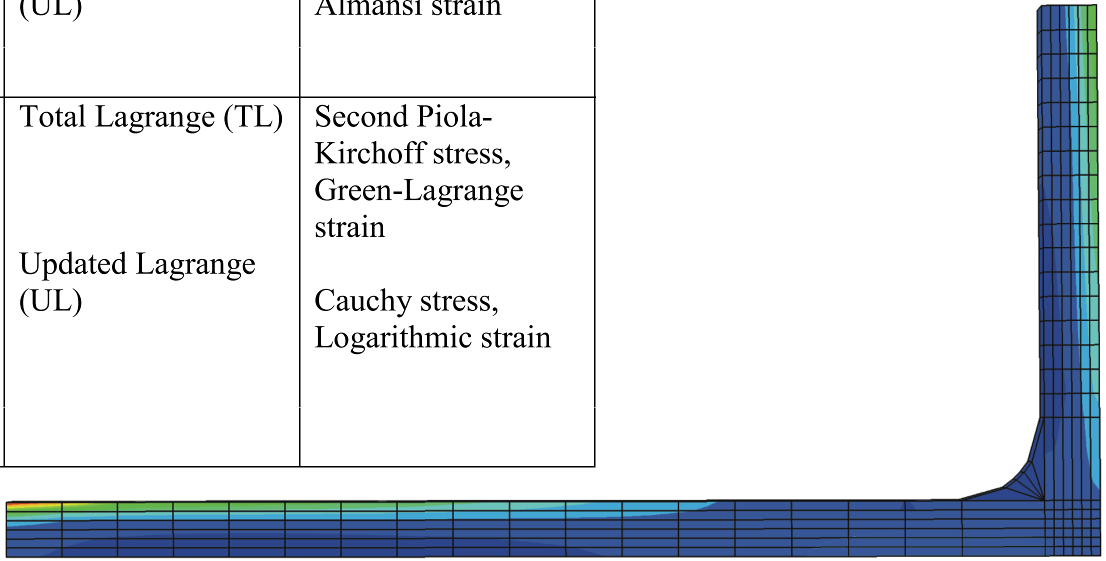
boundary constriants are constant



Introduction to non-linear analysis

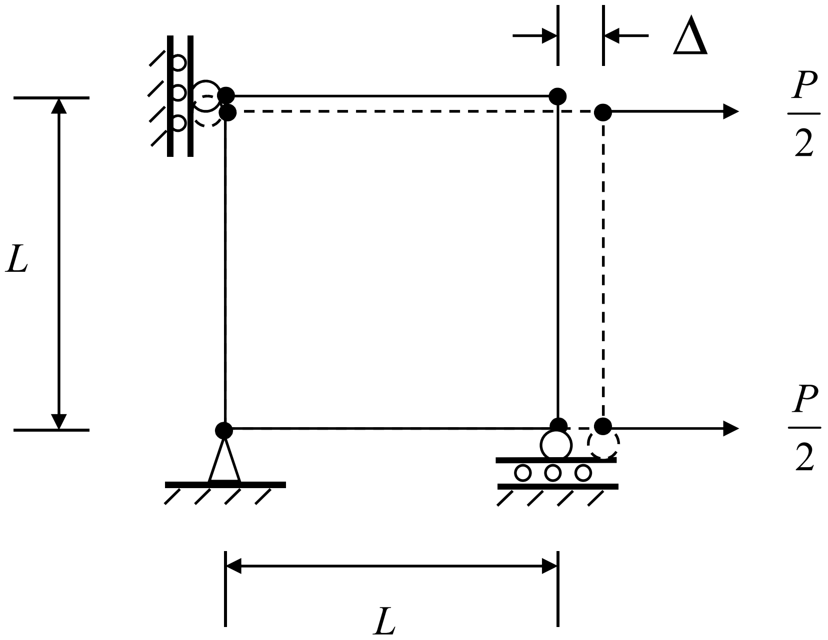
- ## Classification of non-linear analysis

Type of analysis	Description	Typical formulation used	Stress and strain measures used
Materially-nonlinear only	Infinitesimal displacements and strains; stress train relation is non-linear	Materially-nonlinear-only (MNO)	Engineering strain and stress
Large displacements, large rotations but small strains	Displacements and rotations of fibers are large; but fiber extensions and angle changes between fibers are small; stress strain relationship may be linear or non-linear	Total Lagrange (TL) Updated Lagrange (UL)	Second Piola-Kirchoff stress, Green-Lagrange strain Cauchy stress, Almansi strain
Large displacements, large rotations and large strains	Displacements and rotations of fibers are large; fiber extensions and angle changes between fibers may also be large; stress strain relationship may be linear or non-linear	Total Lagrange (TL) Updated Lagrange (UL)	Second Piola-Kirchoff stress, Green-Lagrange strain Cauchy stress, Logarithmic strain

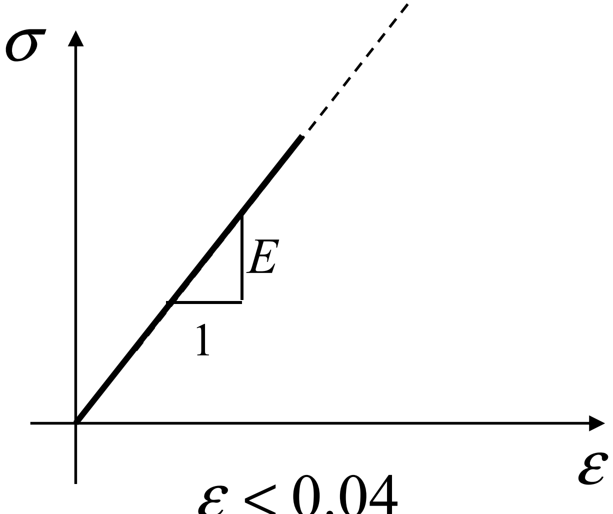


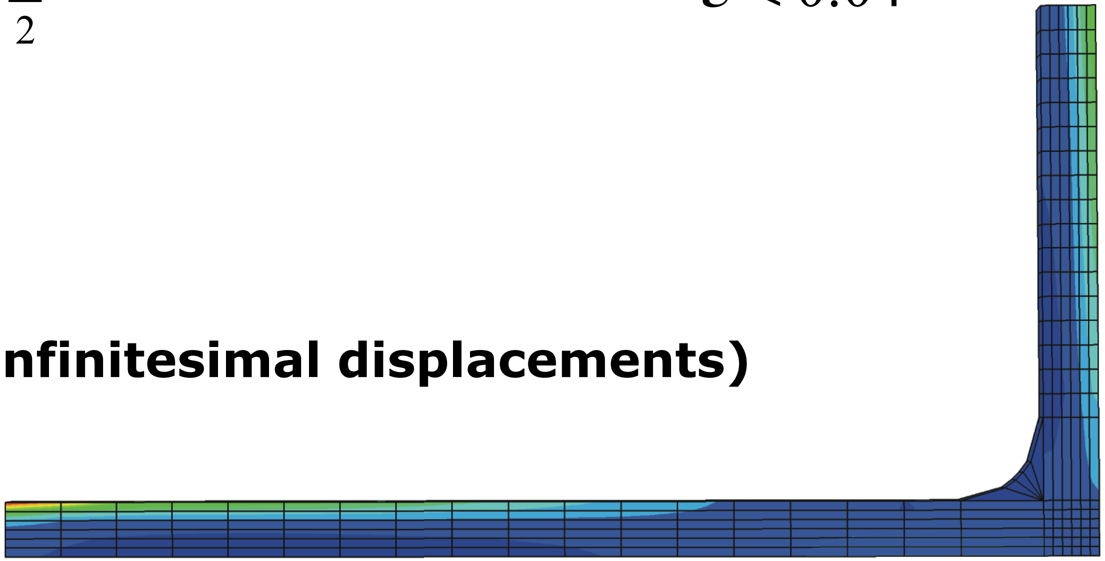
Introduction to non-linear analysis

- ## Classification of non-linear analysis



$$\boxed{\begin{aligned}\sigma &= P/A\\ \varepsilon &= \sigma/E\\ \Delta &= \varepsilon L\end{aligned}}$$

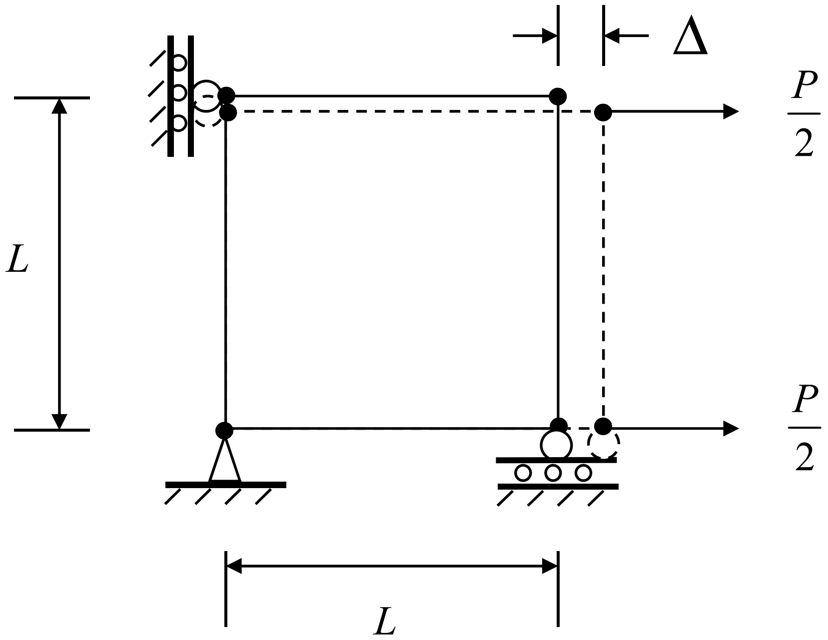




Linear elastic (infinitesimal displacements)

Introduction to non-linear analysis

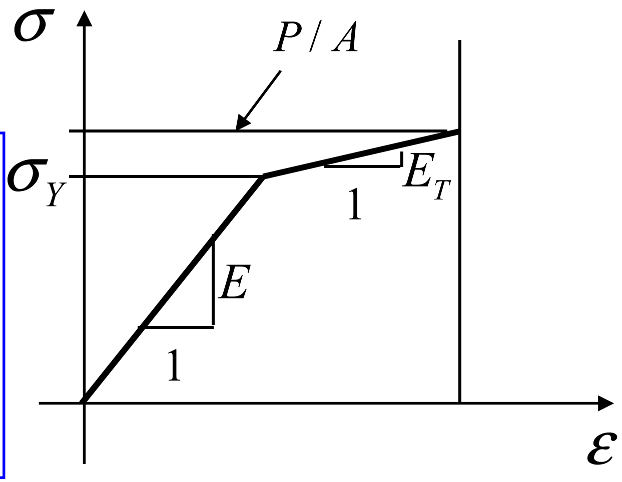
Classification of non-linear analysis

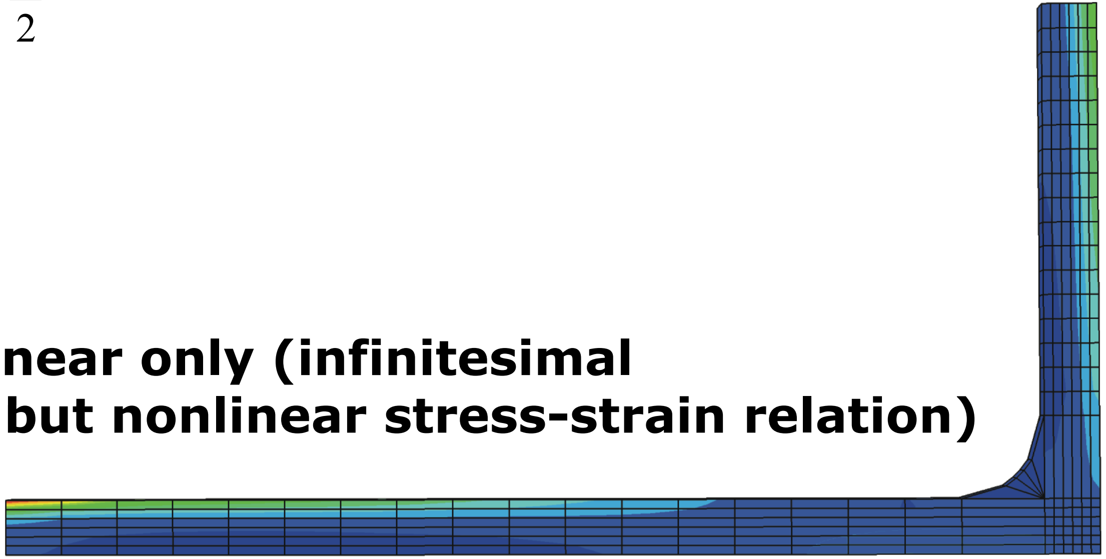


$$\sigma = P/A$$

$$\varepsilon = \frac{\sigma_Y}{E} + \frac{\sigma - \sigma_Y}{E_T}$$

$$\varepsilon < 0.04$$

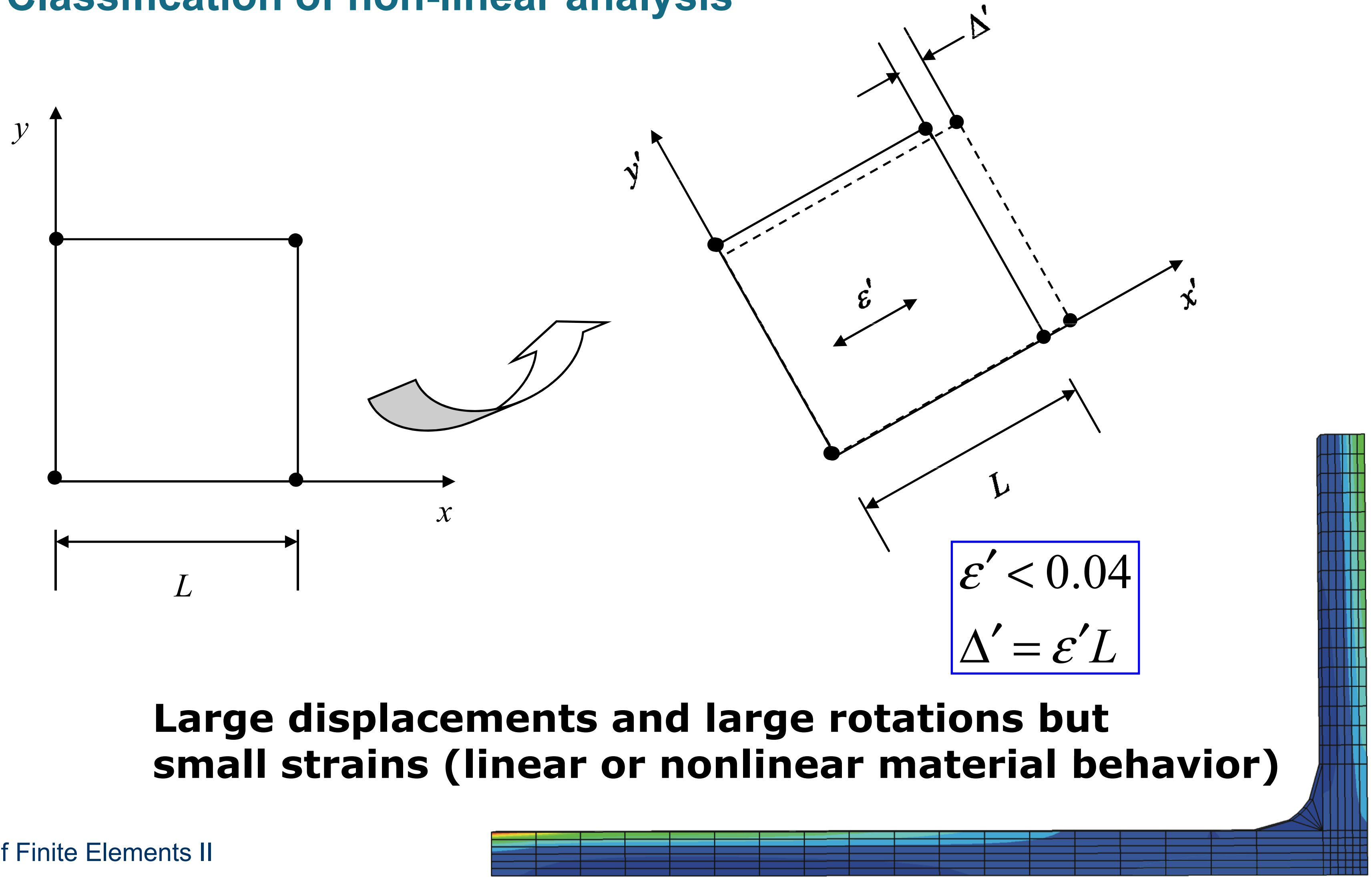




Materially nonlinear only (infinitesimal displacements, but nonlinear stress-strain relation)

Introduction to non-linear analysis

- ## Classification of non-linear analysis



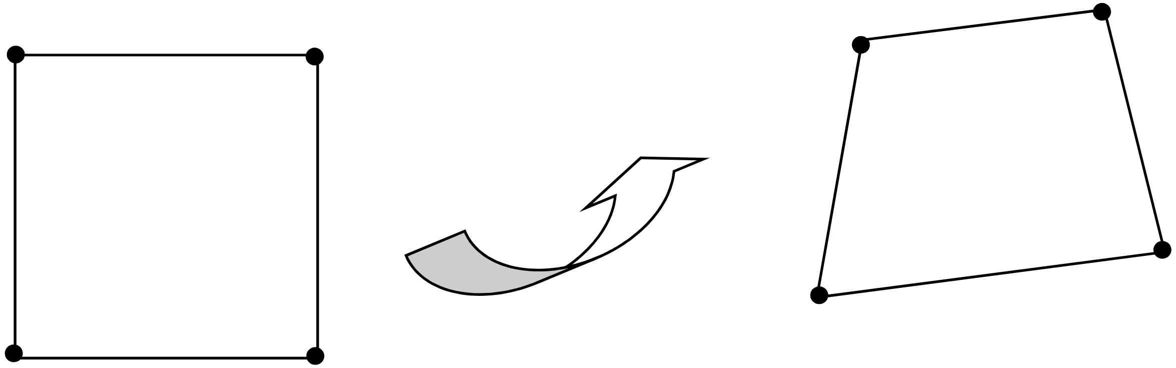
$$\varepsilon' < 0.04$$

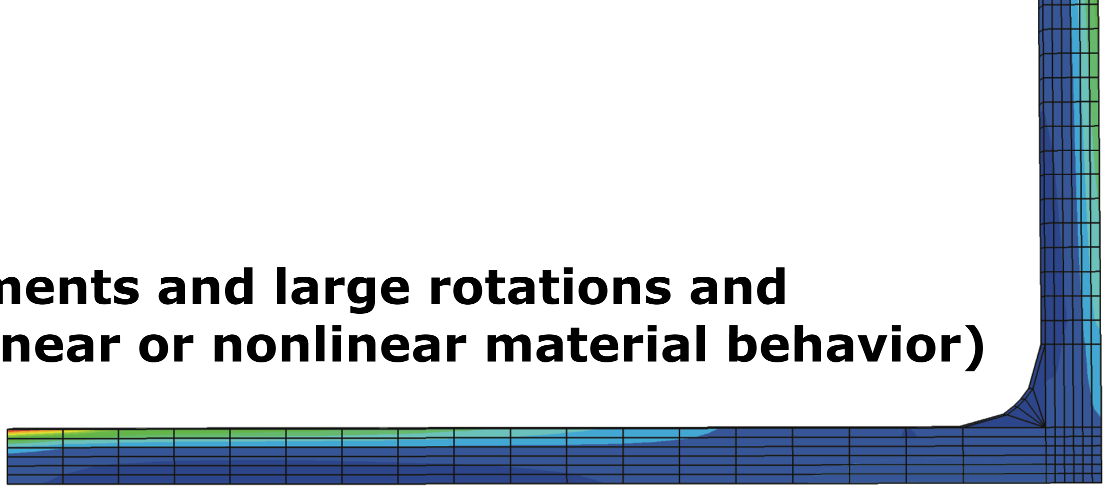
$$\Delta' = \varepsilon' L$$

Large displacements and large rotations but small strains (linear or nonlinear material behavior)

Introduction to non-linear analysis

- ## Classification of non-linear analysis

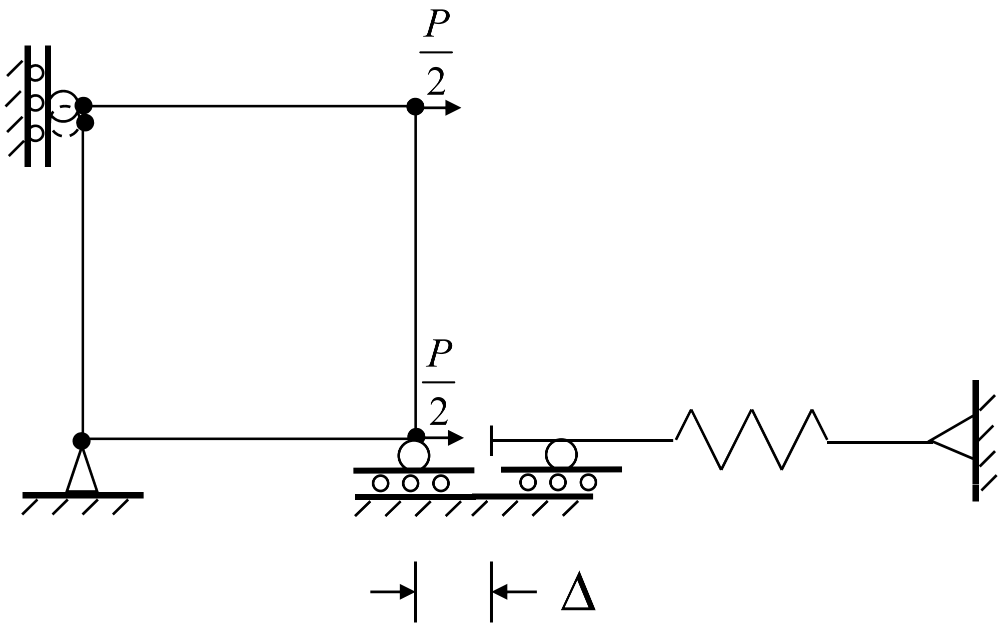




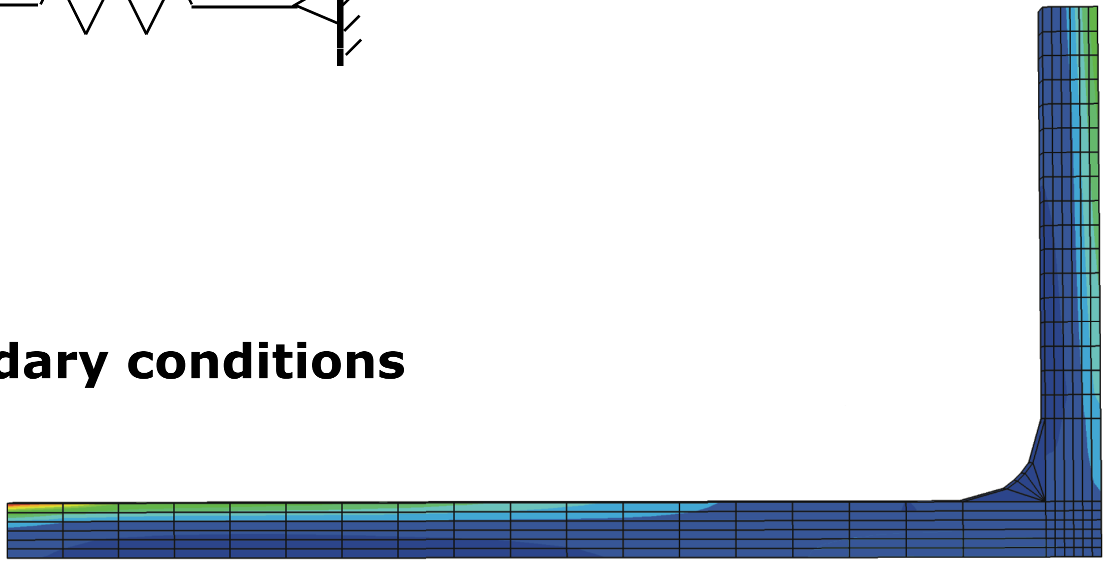
Large displacements and large rotations and large strains (linear or nonlinear material behavior)

Introduction to non-linear analysis

- ## Classification of non-linear analysis

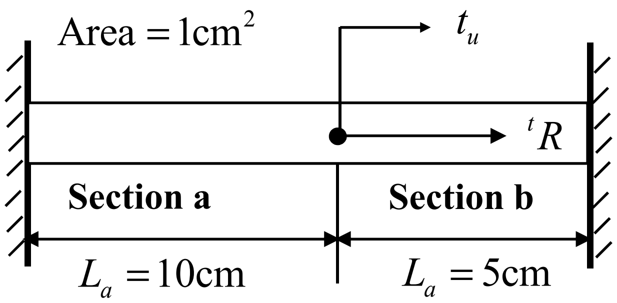


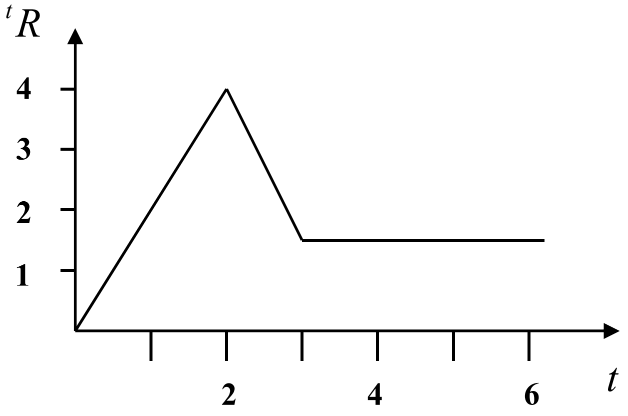
Changing boundary conditions



Introduction to non-linear analysis

- ## Example: Simple bar structure



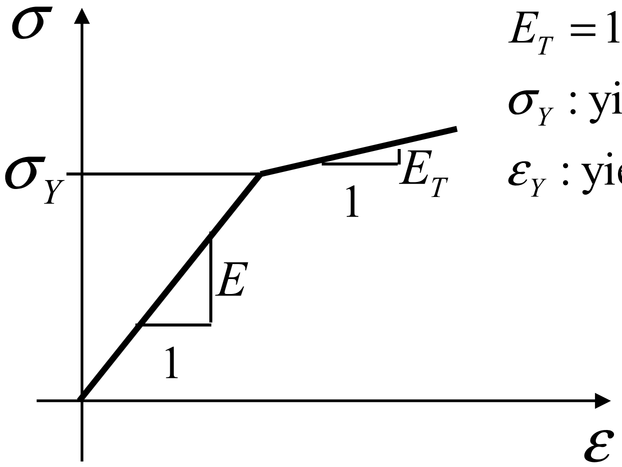


$E = 10^7\,\text{N/cm}^2$

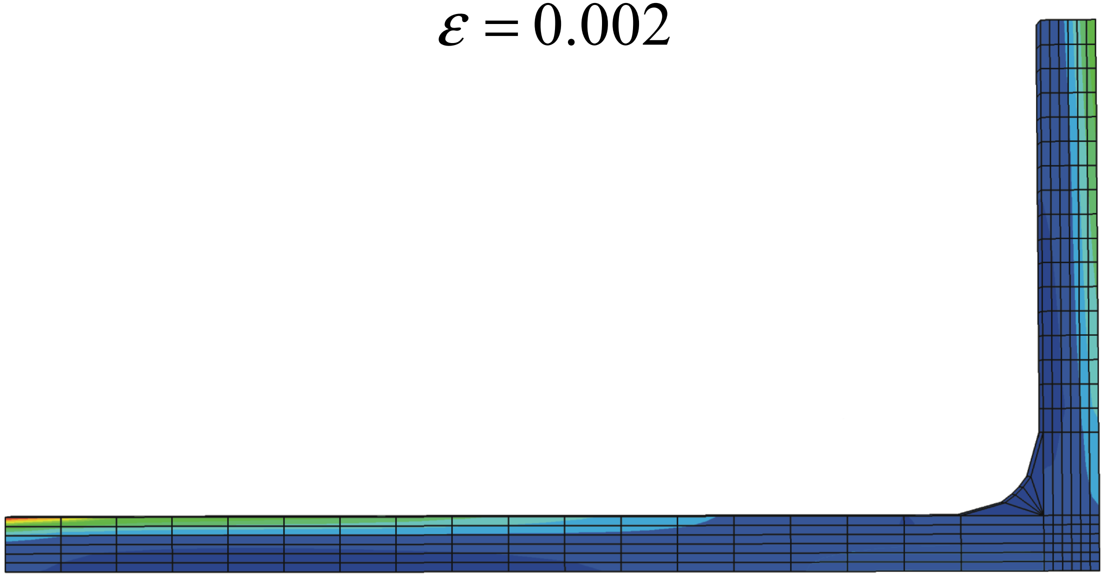
$E_T = 10^5\,\text{N/cm}^2$

σ_Y : yield stress

ε_Y : yield strain

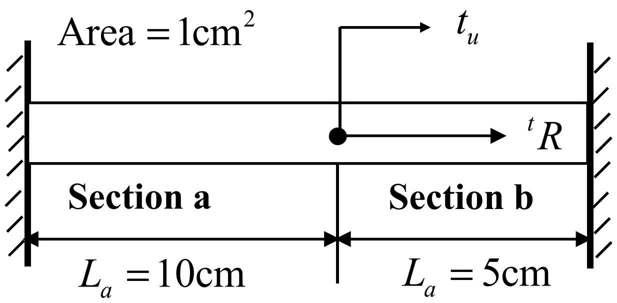


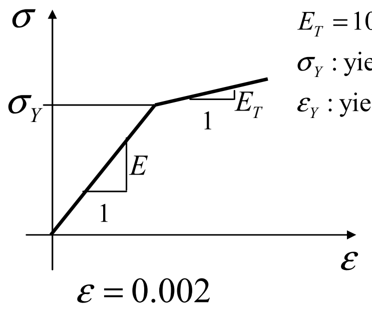
$\varepsilon = 0.002$



Introduction to non-linear analysis

- ## Example: Simple bar structure





$E = 10^7\,\text{N}/\text{cm}^2$
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σ_Y : yield stress
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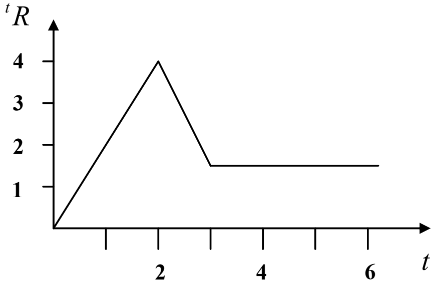


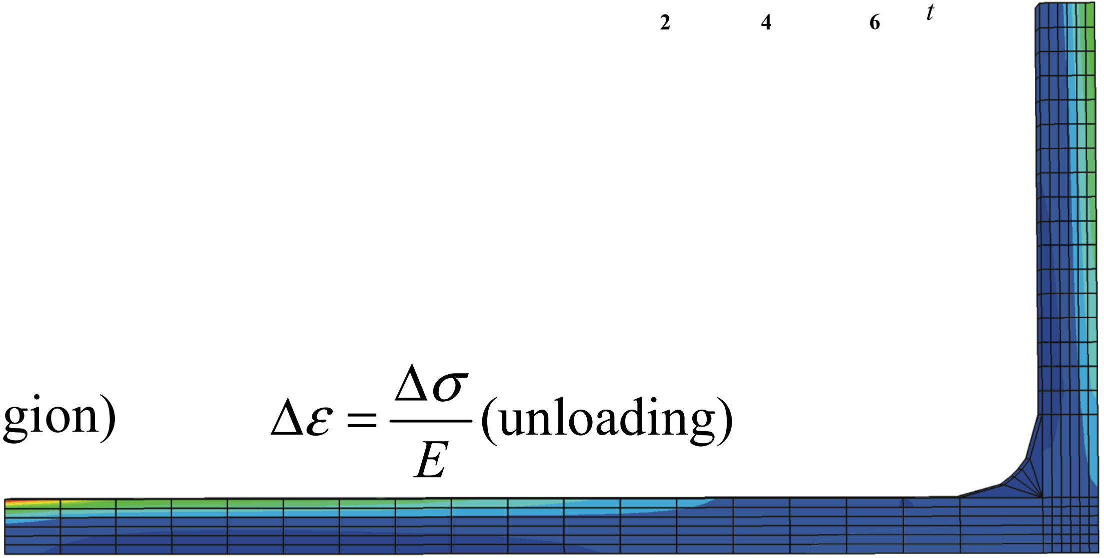
$$ {}^t\varepsilon_a = \frac{{}^tu}{L_a},\ {}^t\varepsilon_b = -\frac{{}^tu}{L_b} $$

$$ {}^tR + {}^t\sigma_b A = {}^t\sigma_a A $$

$$ {}^t\varepsilon = \frac{{}^t\sigma}{E} \text{ (elastic region)} $$

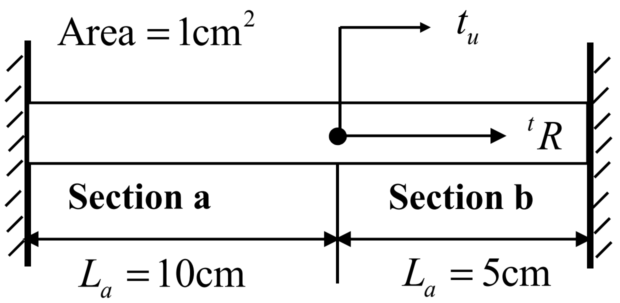
$$ {}^t\varepsilon = \varepsilon_Y + \frac{{}^t\sigma - \sigma_Y}{E_T} \text{ (plastic region)} $$

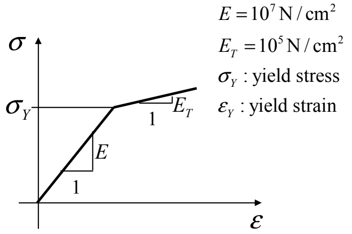
$$ \Delta\varepsilon = \frac{\Delta\sigma}{E} \text{(unloading)} $$

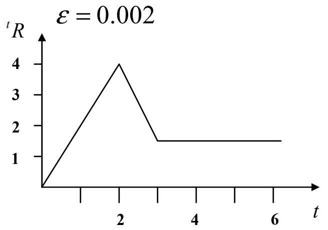


Introduction to non-linear analysis

- ## Example: Simple bar structure







$$ {}^t\varepsilon_a = \frac{{}^tu}{L_a}, {}^t\varepsilon_b = -\frac{{}^tu}{L_b} $$

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$$ {}^t\varepsilon = \frac{{}^t\sigma}{E} \text{ (elastic region)} $$

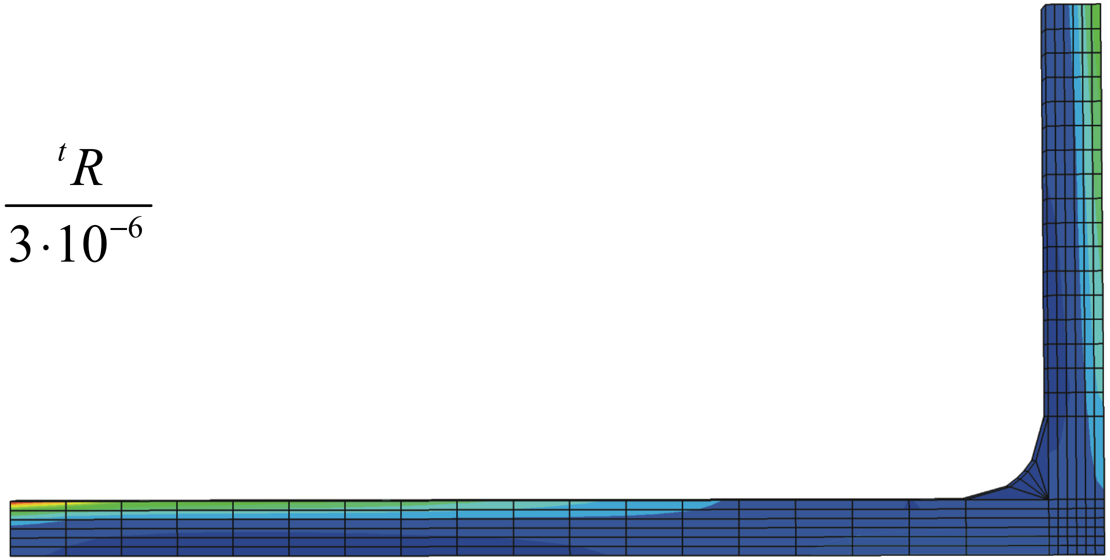
$$ {}^t\varepsilon = \varepsilon_Y + \frac{{}^t\sigma - \sigma_Y}{E_T} \text{ (plastic region)} $$

$$ \Delta\varepsilon = \frac{\Delta\sigma}{E} \text{(unloading)} $$

Both sections elastic

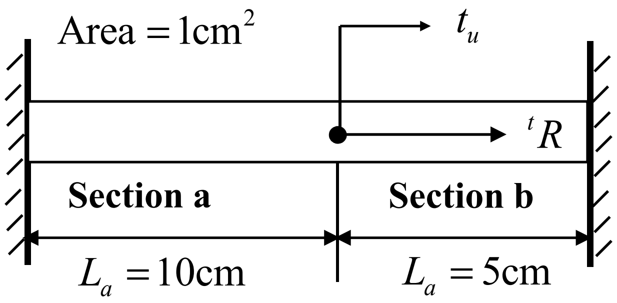
$$ {}^tR = EA\,{}^tu\left(\frac{1}{L_a} + \frac{1}{L_b}\right) \Rightarrow {}^tu = \frac{{}^tR}{3 \cdot 10^{-6}} $$

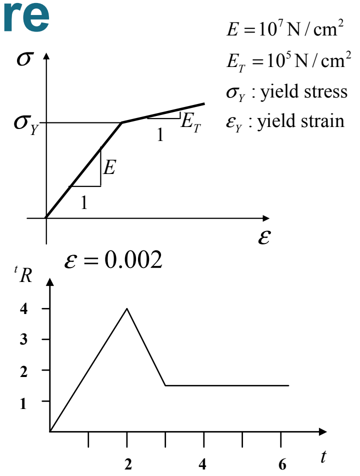
$$ \sigma_a = \frac{{}^tR}{3A}, \sigma_b = -\frac{2}{3}\frac{{}^tR}{A} $$



Introduction to non-linear analysis

Example: Simple bar structure





$$ {}^t\varepsilon_a = \frac{{}^tu}{L_a},\ {}^t\varepsilon_b = -\frac{{}^tu}{L_b} $$

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$$ {}^t\varepsilon = \frac{{}^t\sigma}{E} \ \text{(elastic region)} $$

$$ {}^t\varepsilon = \varepsilon_Y + \frac{{}^t\sigma - \sigma_Y}{E_T} \ \text{(plastic region)} $$

$$ \Delta\varepsilon = \frac{\Delta\sigma}{E} \text{(unloading)} $$

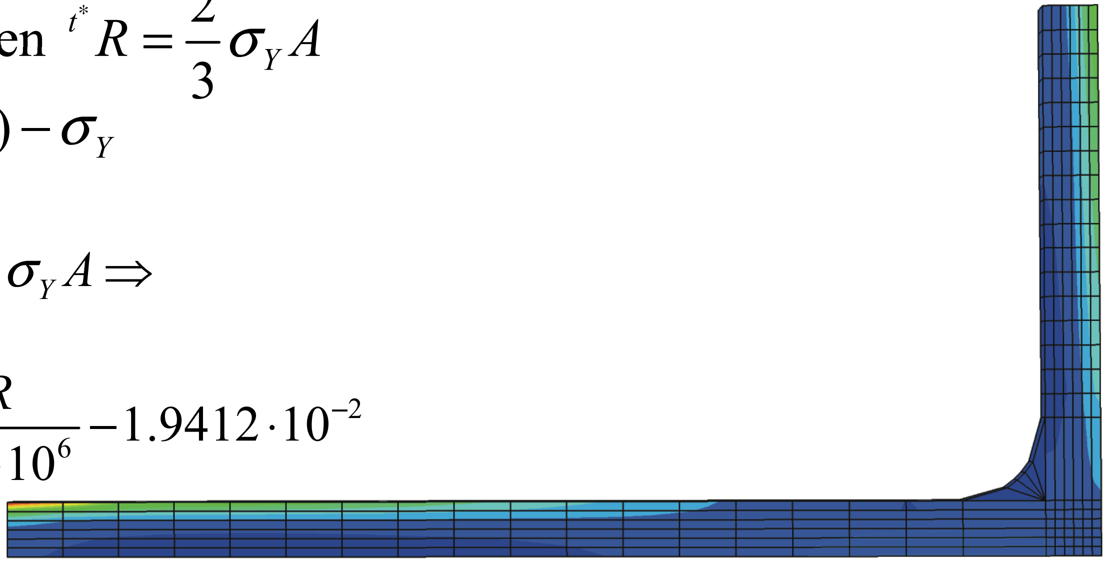
Section a is elastic while section b is plastic

section b will be plastic when $ {}^{t^*}R = \frac{2}{3}\sigma_Y A $

$$ \sigma_a = E\frac{{}^tu}{L_a}, \sigma_b = E_T\left(\frac{{}^tu}{L_b} - \varepsilon_Y\right) - \sigma_Y $$

$$ {}^tR = \frac{EA\,{}^tu}{L_a} + \frac{E_T A\,{}^tu}{L_b} - E_T\varepsilon_Y A + \sigma_Y A \Rightarrow $$

$$ {}^tu = \frac{{}^tR/A + E_T\varepsilon_Y - \sigma_Y}{E/L_a + E/L_b} = \frac{{}^tR}{1.02\cdot 10^6} - 1.9412\cdot 10^{-2} $$



Introduction to non-linear analysis

- **What did we learn from the example?**

The basic problem in general nonlinear analysis is to find a state of equilibrium between externally applied loads and element nodal forces

$$ {}^{t}\mathbf{R} - {}^{t}\mathbf{F} = 0 $$

$$ {}^{t}\mathbf{R} = {}^{t}\mathbf{R}_B + {}^{t}\mathbf{R}_S + {}^{t}\mathbf{R}_C $$

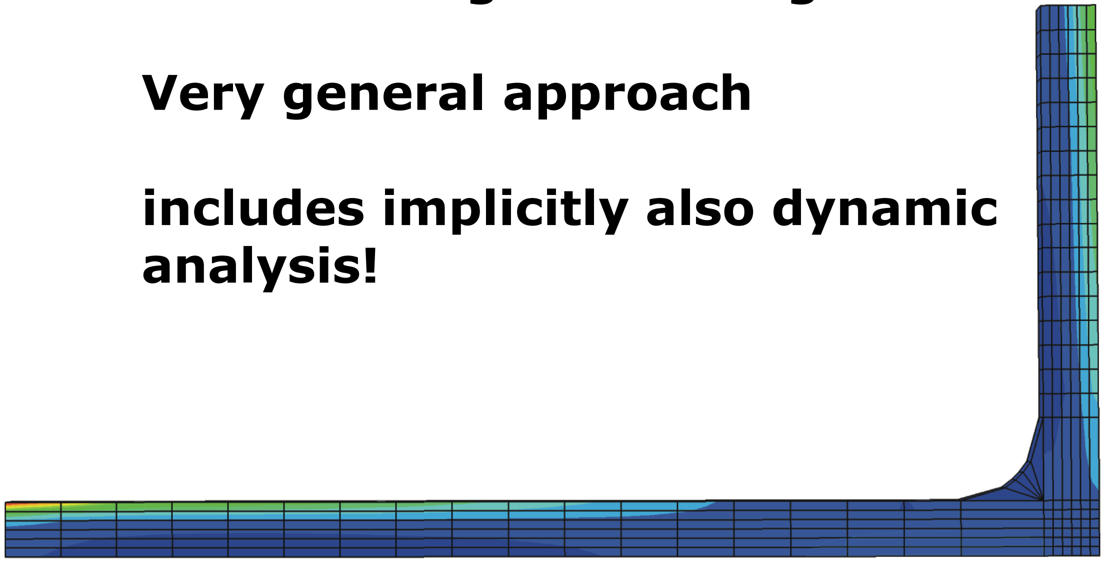
$$ {}^{t}\mathbf{F} = {}^{t}\mathbf{R}_I $$

$$ {}^{t}\mathbf{F} = \sum_{m} \int_{{}^{t}V^{(m)}} {}^{t}\mathbf{B}^{(m)T}\, {}^{t}\tau^{(m)}\, {}^{t}dV^{(m)} $$

We must achieve equilibrium for all time steps when incrementing the loading

Very general approach

includes implicitly also dynamic analysis!



Introduction to non-linear analysis

- **The basic approach in incremental anaylsis is**

$$^{t+\Delta t}\mathbf{R} - {}^{t+\Delta t}\mathbf{F} = 0$$

assuming that $^{t+\Delta t}\mathbf{R}$ is independent of the deformations we have

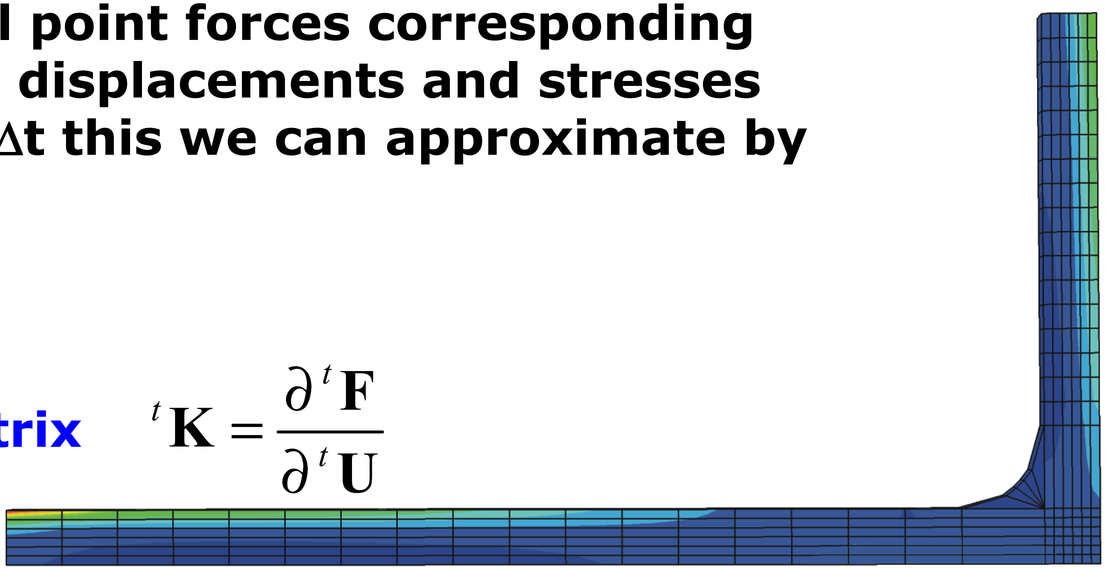
$$^{t+\Delta t}\mathbf{F} = {}^{t}\mathbf{F} + \mathbf{F}$$

We know the solution ^{t}F at time *t* and F is the increment in the nodal point forces corresponding to an increment in the displacements and stresses from time t to time t+Δt this we can approximate by

$$\mathbf{F} = {}^{t}\mathbf{K}\mathbf{U}$$

Tangent stiffness matrix

$$^{t}\mathbf{K} = \frac{\partial\, {}^{t}\mathbf{F}}{\partial\, {}^{t}\mathbf{U}}$$



Introduction to non-linear analysis

- The basic approach in incremental anaylsis is

We may now substitute the tangent stiffness matrix into the equlibrium relation

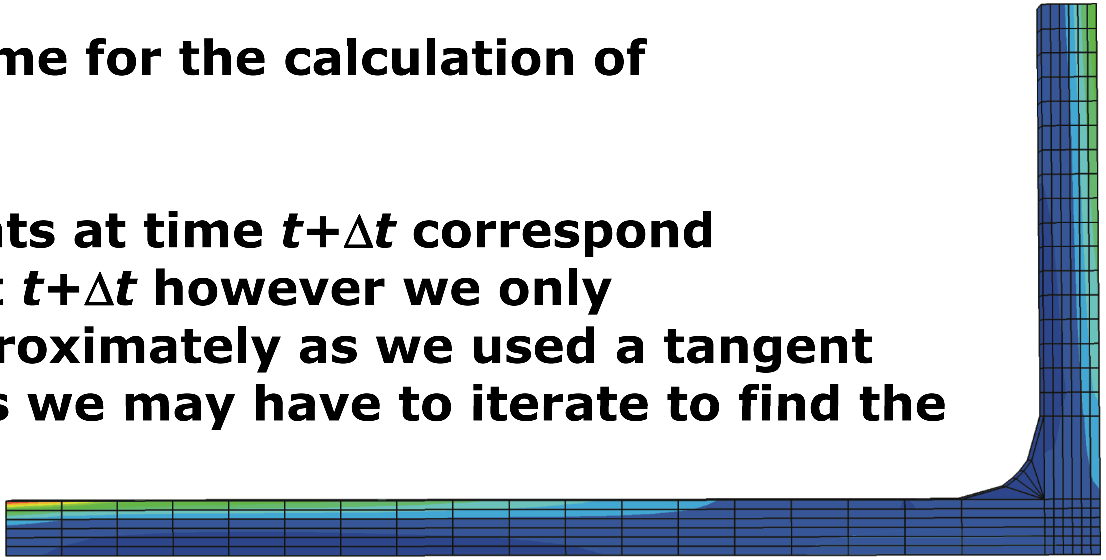
$$
{}^{t}\mathbf{K}\mathbf{U} = {}^{t+\Delta t}\mathbf{R} - {}^{t}\mathbf{F}
$$

$$
\Downarrow
$$

$$
{}^{t+\Delta t}\mathbf{U} = {}^{t}\mathbf{U} + \mathbf{U}
$$

which gives us a scheme for the calculation of the displacements

the exact displacements at time *t*+Δ*t* correspond to the applied loads at *t*+Δ*t* however we only determined these approximately as we used a tangent stiffness matrix – thus we may have to iterate to find the solution



Introduction to non-linear analysis

- The basic approach in incremental anaylsis is

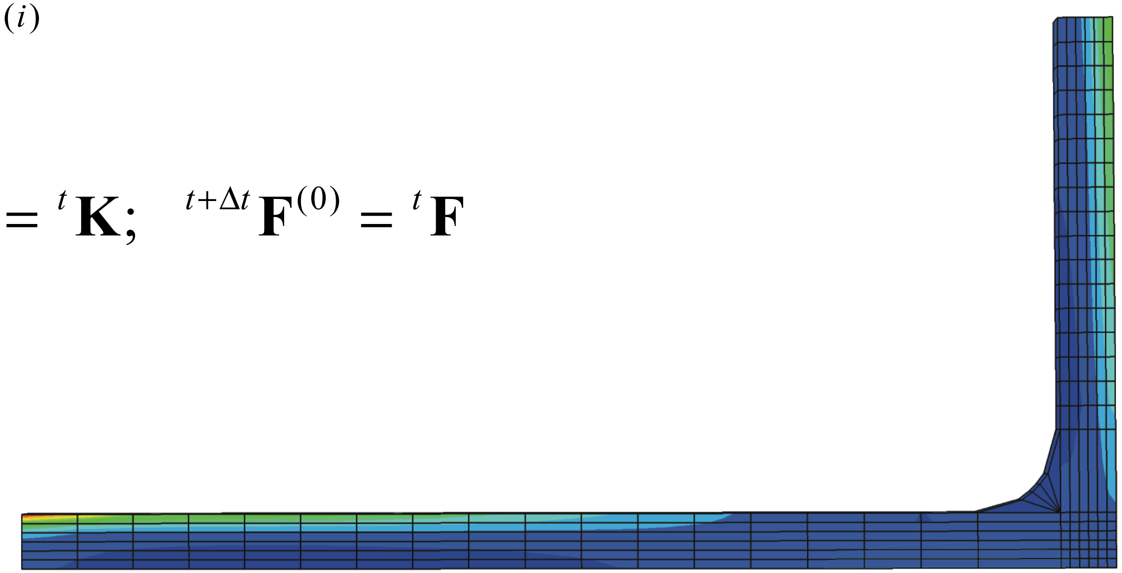
We may use the Newton-Raphson iteration scheme to find the equlibrium within each load increment

$$ {}^{t+\Delta t}\mathbf{K}^{(i-1)}\Delta\mathbf{U}^{(i)} = {}^{t+\Delta t}\mathbf{R} - {}^{t+\Delta t}\mathbf{F}^{(i-1)} $$ **(out of balance load vector)**

$$ {}^{t+\Delta t}\mathbf{U}^{(i)} = {}^{t+\Delta t}\mathbf{U}^{(i-1)} + \Delta\mathbf{U}^{(i)} $$

with initial conditions

$$ {}^{t+\Delta t}\mathbf{U}^{(0)} = {}^{t}\mathbf{U}; \quad {}^{t+\Delta t}\mathbf{K}^{(0)} = {}^{t}\mathbf{K}; \quad {}^{t+\Delta t}\mathbf{F}^{(0)} = {}^{t}\mathbf{F} $$



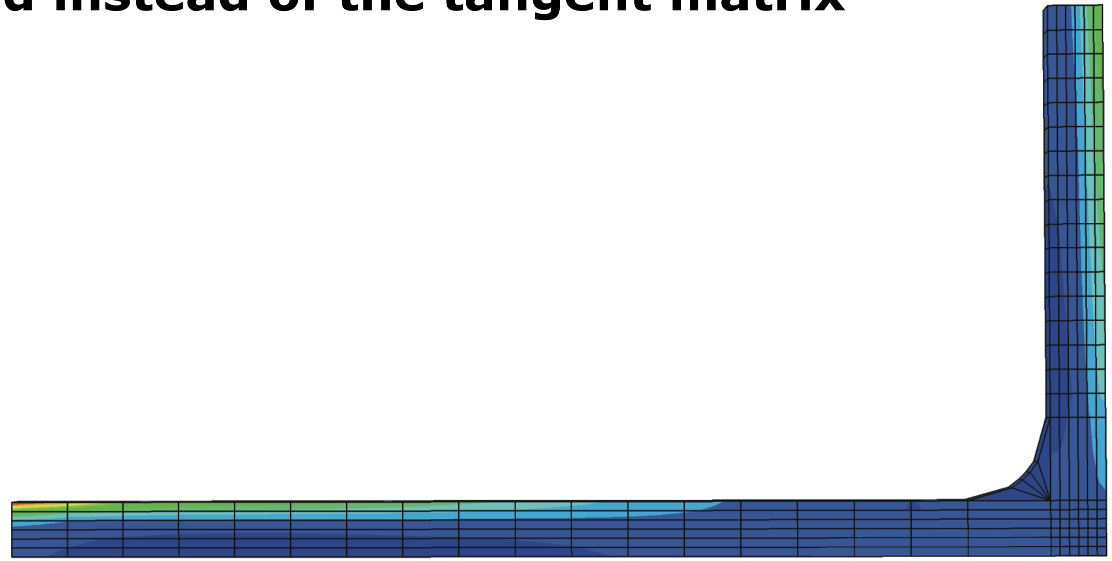
Introduction to non-linear analysis

- The basic approach in incremental anaylsis is

It may be expensive to calculate the tangent stiffness matrix and;

in the Modified Newton-Raphson iteration scheme it is thus only calculated in the beginning of each new load step

in the quasi-Newton iteration schemes the secant stiffness matrix is used instead of the tangent matrix



The continuum mechanics incremental equations

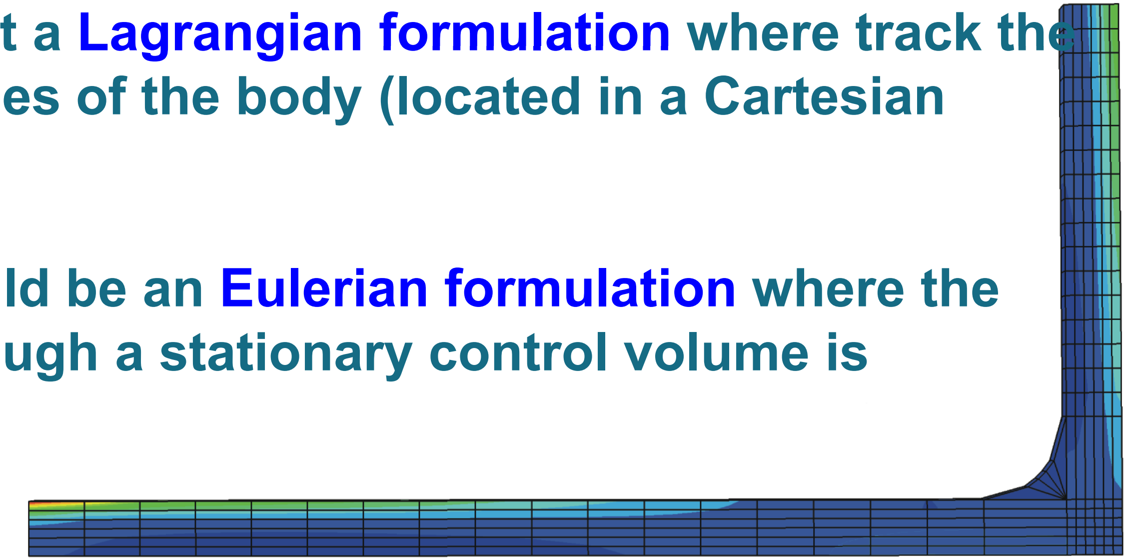
- The basic problem:

  We want to establish the solution using an incremental formulation

  The equilibrium must be established for the considered body in ist current configuration

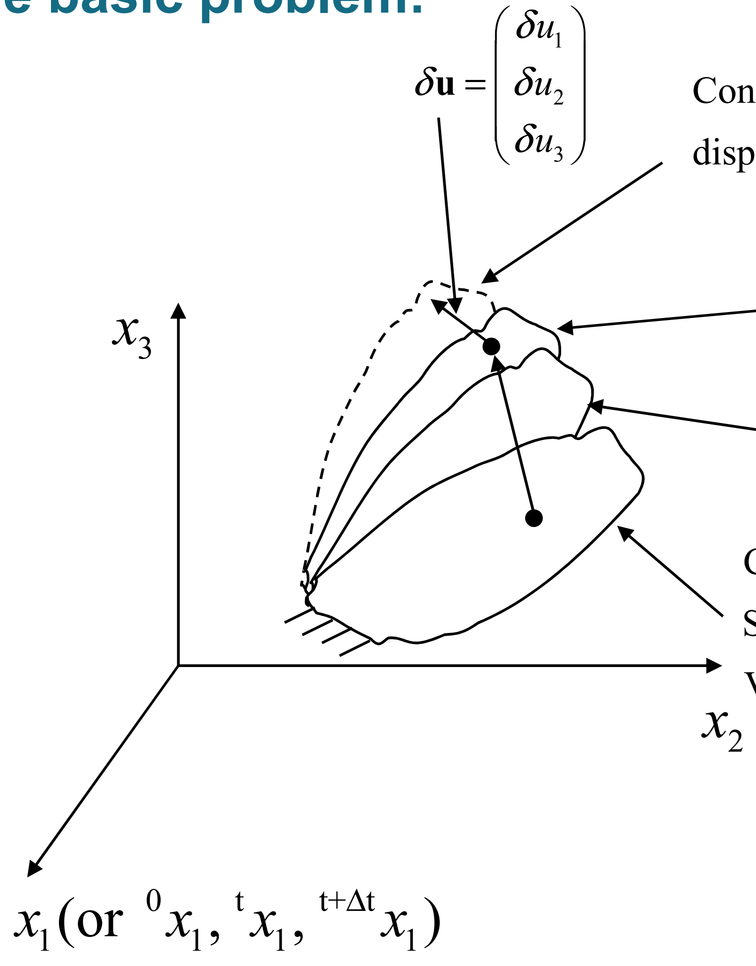
  In proceeding we adopt a **Lagrangian formulation** where track the movement of all particles of the body (located in a Cartesian coordinate system)

  Another approach would be an **Eulerian formulation** where the motion of material through a stationary control volume is considered



The continuum mechanics incremental equations

- **The basic problem:**



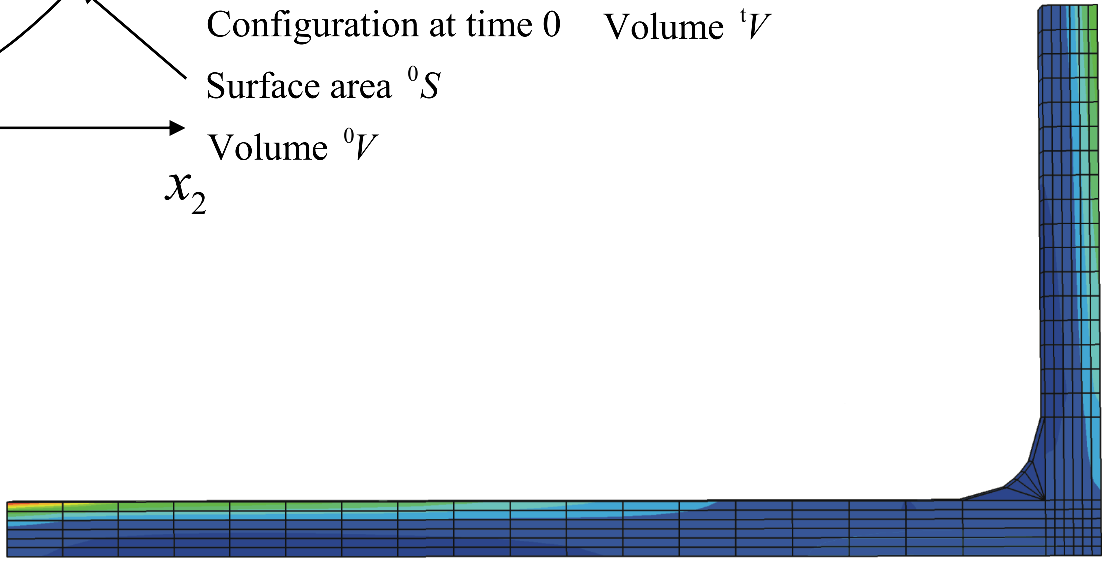
$$\delta \mathbf{u} = \begin{pmatrix} \delta u_1 \\ \delta u_2 \\ \delta u_3 \end{pmatrix}$$

Configuration corresponding to variation in displacements $\delta \mathbf{u}$ at ${}^{t+\Delta t}\mathbf{u}$

Configuration at time $t+\Delta t$
Surface area ${}^{t+\Delta t}S$
Volume ${}^{t+\Delta t}V$

Configuration at time t
Surface area ${}^{t}S$
Volume ${}^{t}V$

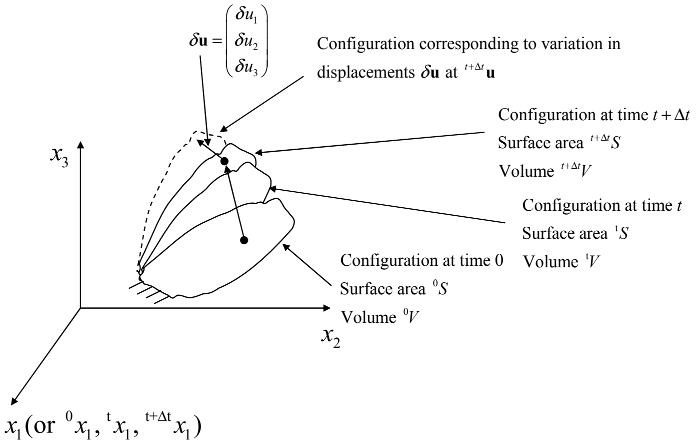
Configuration at time 0
Surface area ${}^{0}S$
Volume ${}^{0}V$



The continuum mechanics incremental equations

- ## The Lagrangian formulation

  We express equilibrium of the body at time *t+Δt* using the principle of virtual displacements



$$\int_{{}^{t+\Delta t}V} {}^{t+\Delta t}\tau \delta_{t+\Delta t} e_{ij} d\,{}^{t+\Delta t}V = {}^{t+\Delta t}R$$

${}^{t+\Delta t}\tau$: Cartesian components of the Cauchy stress tensor

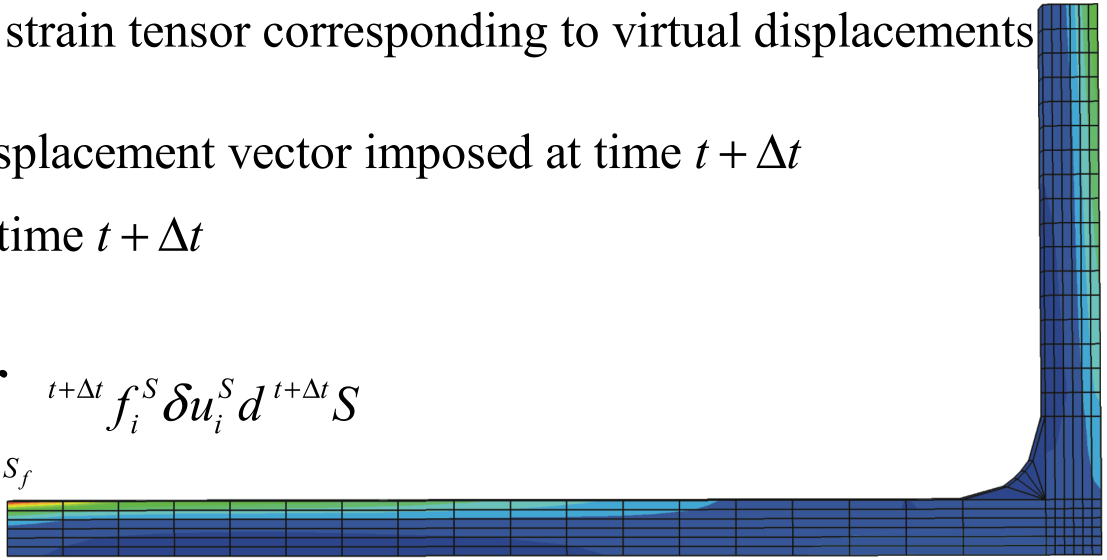
$$\delta_{t+\Delta t} e_{ij} = \frac{1}{2}\left(\frac{\partial \delta u_i}{\partial\, {}^{t+\Delta t}x_j} + \frac{\partial \delta u_j}{\partial\, {}^{t+\Delta t}x_i}\right) = \text{strain tensor corresponding to virtual displacements}$$

δu_i : Components of virtual displacement vector imposed at time $t+\Delta t$

${}^{t+\Delta t}x_i$: Cartesian coordinate at time $t+\Delta t$

${}^{t+\Delta t}V$: Volume at time $t+\Delta t$

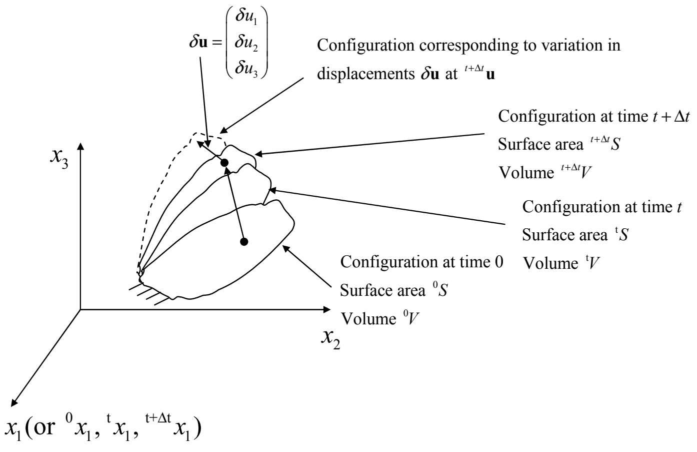
$${}^{t+\Delta t}R = \int_{{}^{t+\Delta t}V} {}^{t+\Delta t}f_i^B \delta u_i d\,{}^{t+\Delta t}V = \int_{{}^{t+\Delta t}S_f} {}^{t+\Delta t}f_i^S \delta u_i^S d\,{}^{t+\Delta t}S$$



The continuum mechanics incremental equations

- **The Lagrangian formulation**

We express equilibrium of the body at time *t+Δt* using the principle of virtual displacements



$$ {}^{t+\Delta t}R = \int_{{}^{t+\Delta t}V} {}^{t+\Delta t}f_i^B \, \delta u_i \, d\,{}^{t+\Delta t}V = \int_{{}^{t+\Delta t}S_f} {}^{t+\Delta t}f_i^S \, \delta u_i^S \, d\,{}^{t+\Delta t}S $$

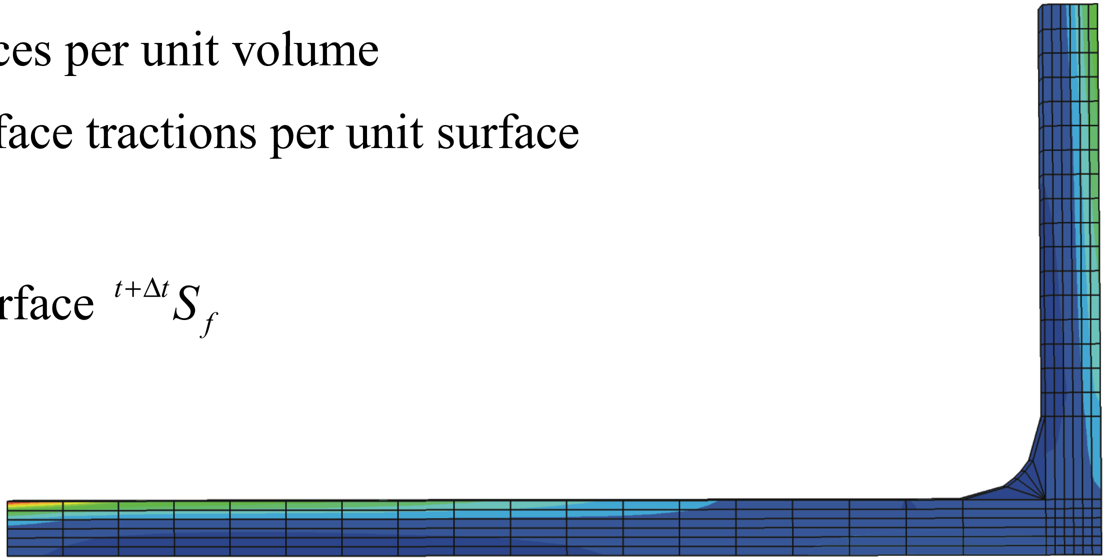
where

${}^{t+\Delta t}f_i^B$: externally apllied forces per unit volume

${}^{t+\Delta t}f_i^S$: externally apllied surface tractions per unit surface

${}^{t+\Delta t}S_f$: surface at time $t+\Delta t$

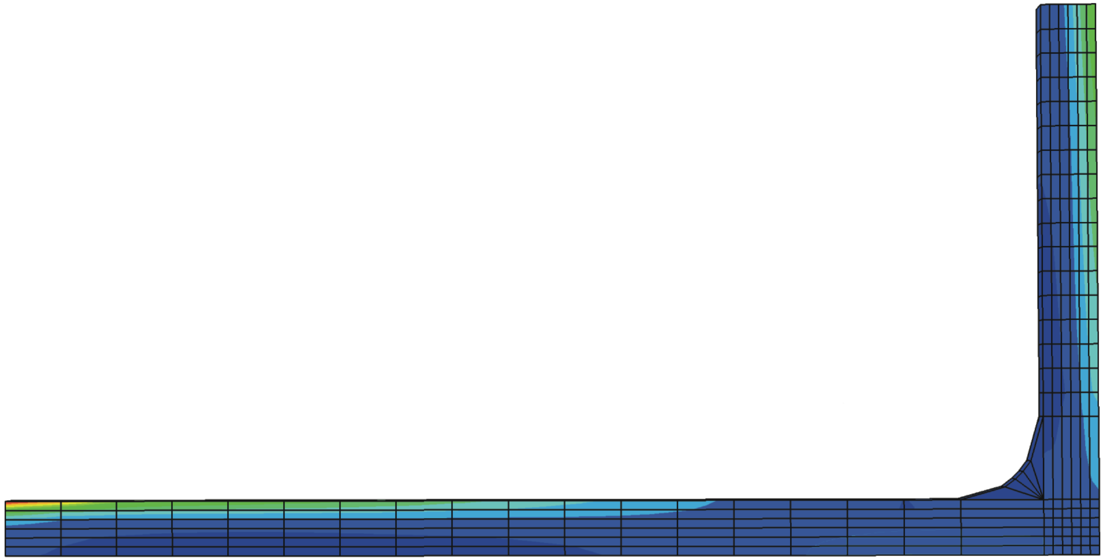
δu_i^S : δu_i evaluated at the surface ${}^{t+\Delta t}S_f$



The continuum mechanics incremental equations

- **The Lagrangian formulation**

  We recognize that our derivations from linear finite element theory are unchanged – but applied to the body in the configuration at time *$t+\Delta t$*



The continuum mechanics incremental equations

- **In the further we introduce an appropriate notation:**

Coordinates and displacements are related as:

$$ {}^{t}x_i = {}^{0}x_i + {}^{t}u_i $$

$$ {}^{t+\Delta t}x_i = {}^{0}x_i + {}^{t+\Delta t}u_i $$

Increments in displacements are related as:

$$ {}_{t}u_i = {}^{t+\Delta t}u_i - {}^{t}u_i $$

Reference configurations are indexed as e.g.:

${}^{t+\Delta t}_{\;0}f_i^S$ where the lower left index indicates the reference configuration

$$ {}^{t+\Delta t}\tau_{ij} = {}^{t+\Delta t}_{t+\Delta t}\tau_{ij} $$

Differentiation is indexed as:

$$ {}^{t+\Delta t}_{\;0}u_{i,j} = \frac{\partial\, {}^{t+\Delta t}u_i}{\partial\, {}^{0}x_j}, \qquad {}^{\;0}_{t+\Delta t}x_{m,n} = \frac{\partial\, {}^{0}x_m}{\partial\, {}^{t+\Delta t}x_n} $$