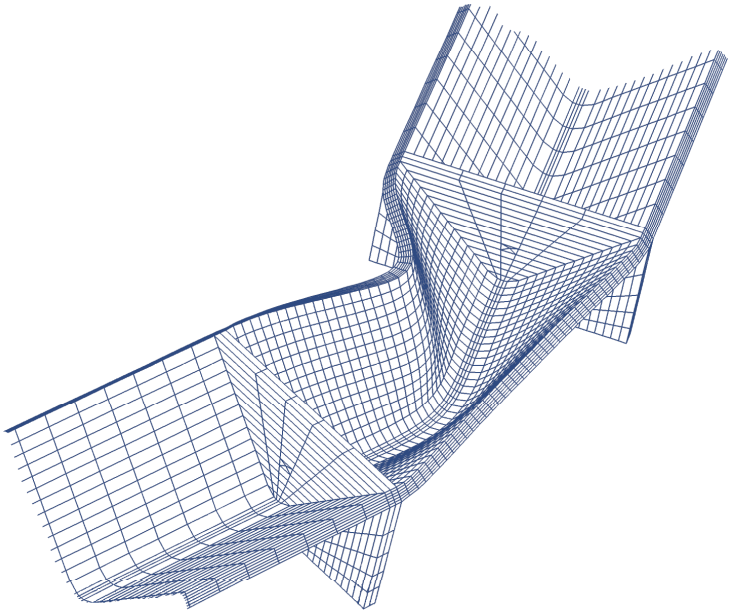
The Finite Element Method for the Analysis of Non-Linear and Dynamic Systems



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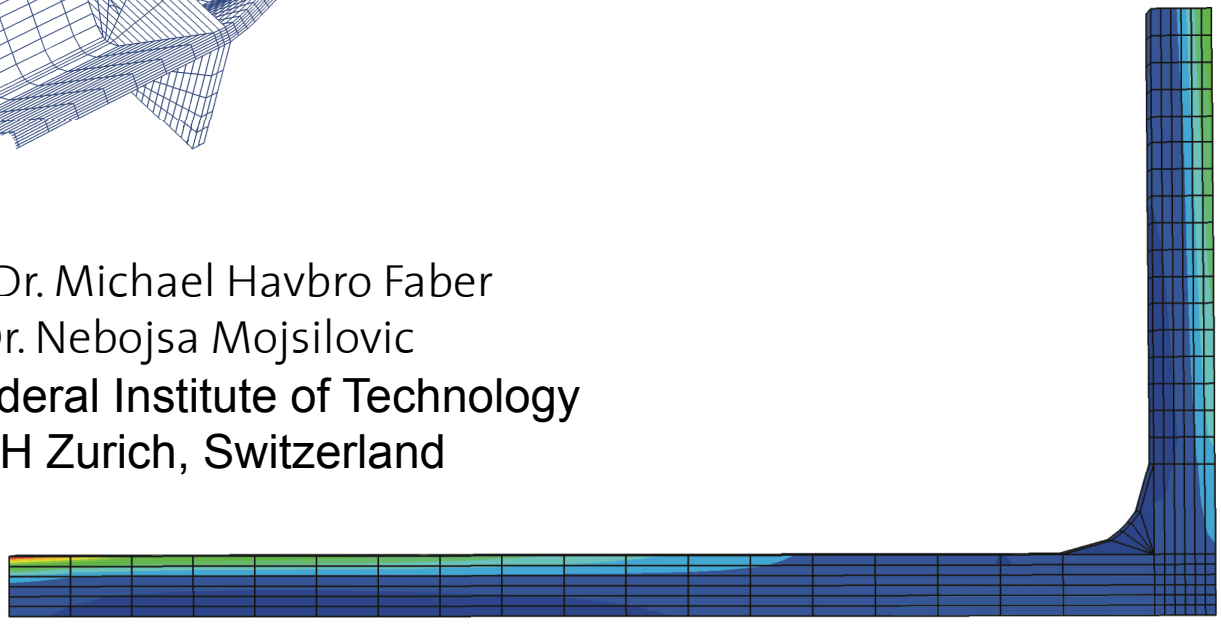


Contents of Today's Lecture

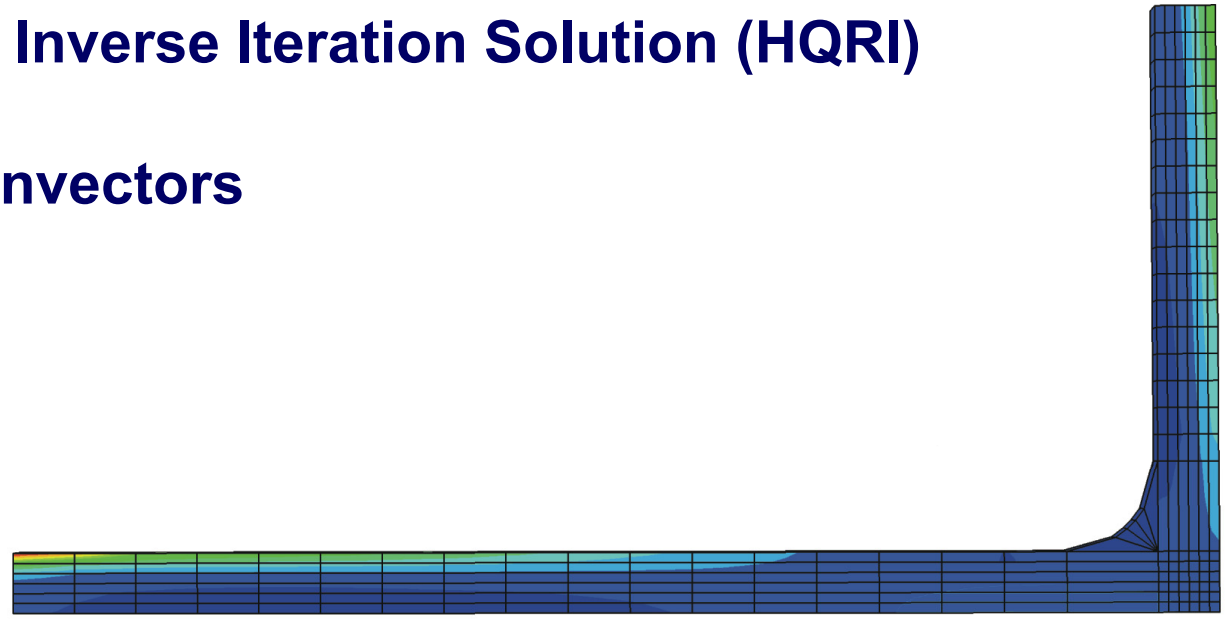
- **Solution of Equilibrium Equations in Dynamic Analysis**

  Transformation Methods

  - The Jacobi Method

  - The Generalized Jacobi Method

  - The Householders QR Inverse Iteration Solution (HQRI)
    - **The QR iteration**
    - **Calculation of eigenvectors**



The Objective - Mode Superposition

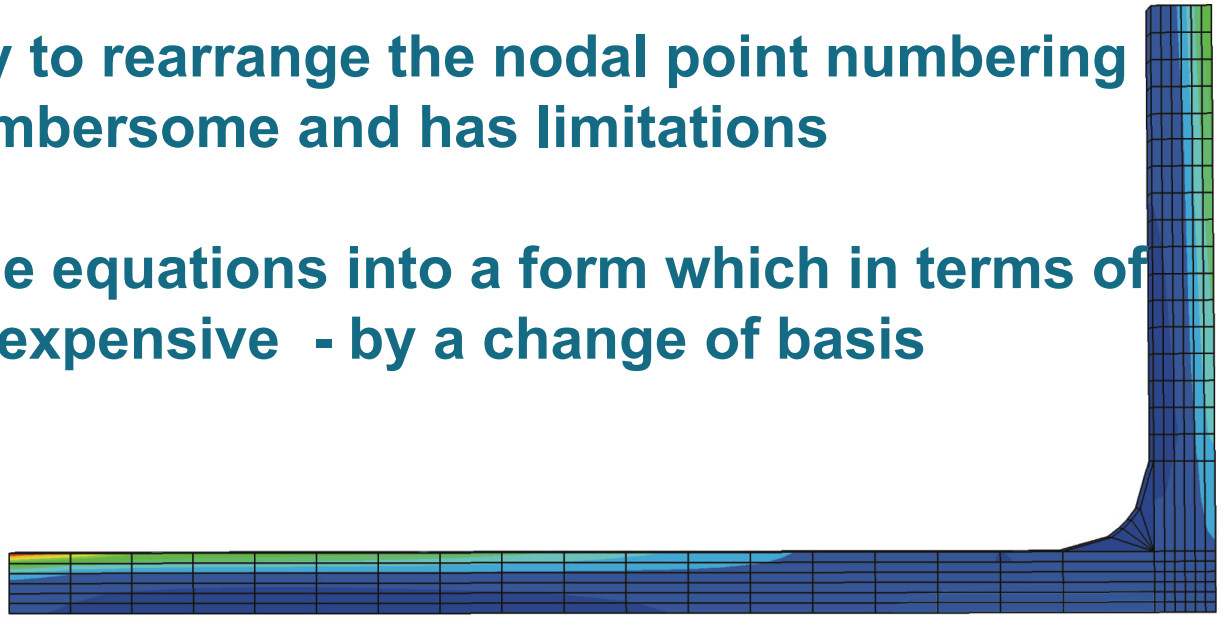
- **Modal Generalized Displacements**

The direct integration methods necessitate that the finite element equations are evaluated for each time step

The bandwidth of the matrixes M, C and K depend on the numbering of the finite element nodal points

In principle we could try to rearrange the nodal point numbering but this approach is cumbersome and has limitations

Instead we transform the equations into a form which in terms of numerical effort is less expensive - by a change of basis



The Objective - Mode Superposition

- **Change of Basis to Modal Coordinates**

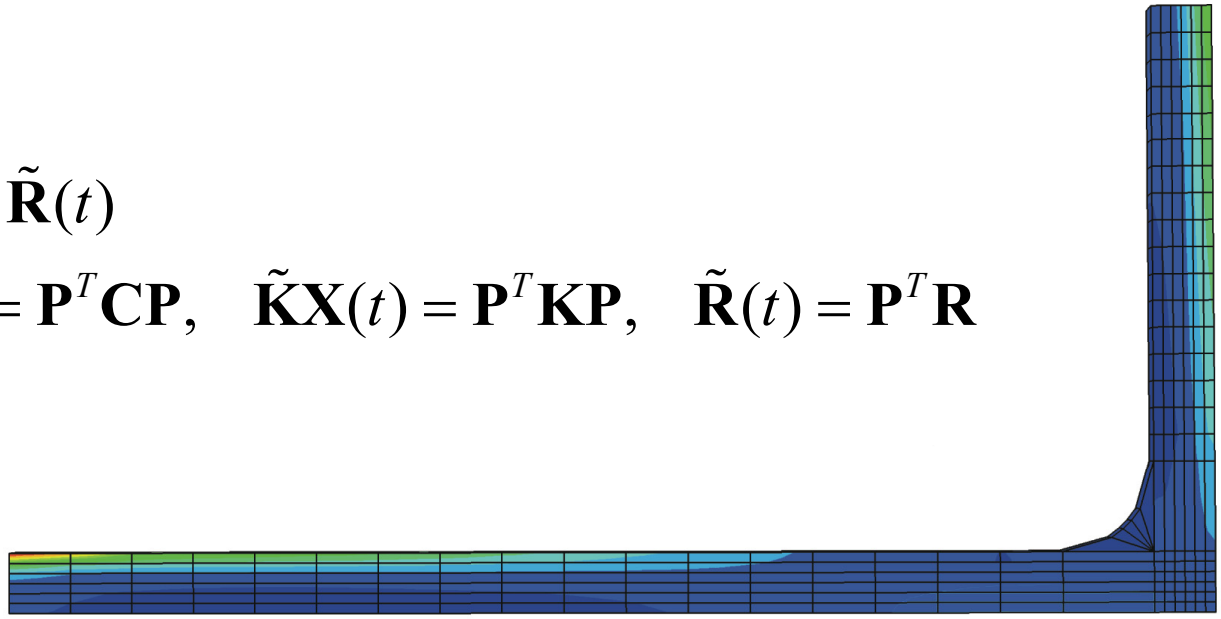
The following transformation is introduced:

$$\mathbf{U}(t) = \mathbf{P}\mathbf{X}(t)$$

P: ***n*** **x** ***n*** **square matrix**
X(*t*): **time dependent vector of order n**

$$\tilde{\mathbf{M}}\ddot{\mathbf{X}}(t) + \tilde{\mathbf{C}}\dot{\mathbf{X}}(t) + \tilde{\mathbf{K}}\mathbf{X}(t) = \tilde{\mathbf{R}}(t)$$

$$\tilde{\mathbf{M}}\ddot{\mathbf{X}}(t) = \mathbf{P}^T\mathbf{M}\mathbf{P}, \quad \tilde{\mathbf{C}}\dot{\mathbf{X}}(t) = \mathbf{P}^T\mathbf{C}\mathbf{P}, \quad \tilde{\mathbf{K}}\mathbf{X}(t) = \mathbf{P}^T\mathbf{K}\mathbf{P}, \quad \tilde{\mathbf{R}}(t) = \mathbf{P}^T\mathbf{R}$$



The Objective - Mode Superposition

$$\tilde{\mathbf{M}}\ddot{\mathbf{X}}(t)+\tilde{\mathbf{C}}\dot{\mathbf{X}}(t)+\tilde{\mathbf{K}}\mathbf{X}(t)=\tilde{\mathbf{R}}(t)$$

$$\tilde{\mathbf{M}}\ddot{\mathbf{X}}(t)=\mathbf{P}^T\mathbf{M}\mathbf{P},\quad \tilde{\mathbf{C}}\dot{\mathbf{X}}(t)=\mathbf{P}^T\mathbf{C}\mathbf{P},\quad \tilde{\mathbf{K}}\mathbf{X}(t)=\mathbf{P}^T\mathbf{K}\mathbf{P},\quad \tilde{\mathbf{R}}(t)=\mathbf{P}^T\mathbf{R}$$

- **Change of Basis to Modal Coordinates**

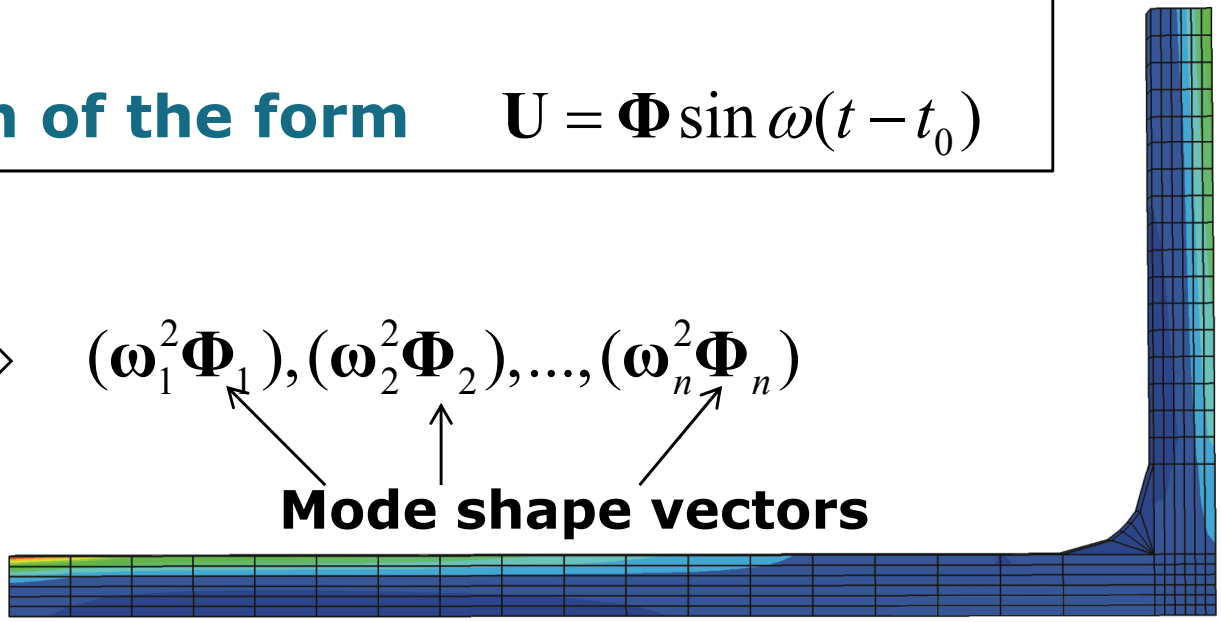
The question is – how to choose P?

A good choice is to take basis in the free vibration solution – neglecting damping, i.e.:

$$\mathbf{M}\ddot{\mathbf{U}}+\mathbf{K}\mathbf{U}=0$$

which has a solution of the form $\mathbf{U}=\boldsymbol{\Phi}\sin\omega(t-t_0)$

$$\mathbf{K}\boldsymbol{\Phi}=\omega^2\mathbf{M}\boldsymbol{\Phi} \Rightarrow (\omega_1^2\boldsymbol{\Phi}_1),(\omega_2^2\boldsymbol{\Phi}_2),\ldots,(\omega_n^2\boldsymbol{\Phi}_n)$$



The Objective - Mode Superposition

- **Change of Basis to Modal Coordinates**

Any of the solutions $(\omega_1^2\Phi_1),(\omega_2^2\Phi_2),\ldots,(\omega_n^2\Phi_n)$

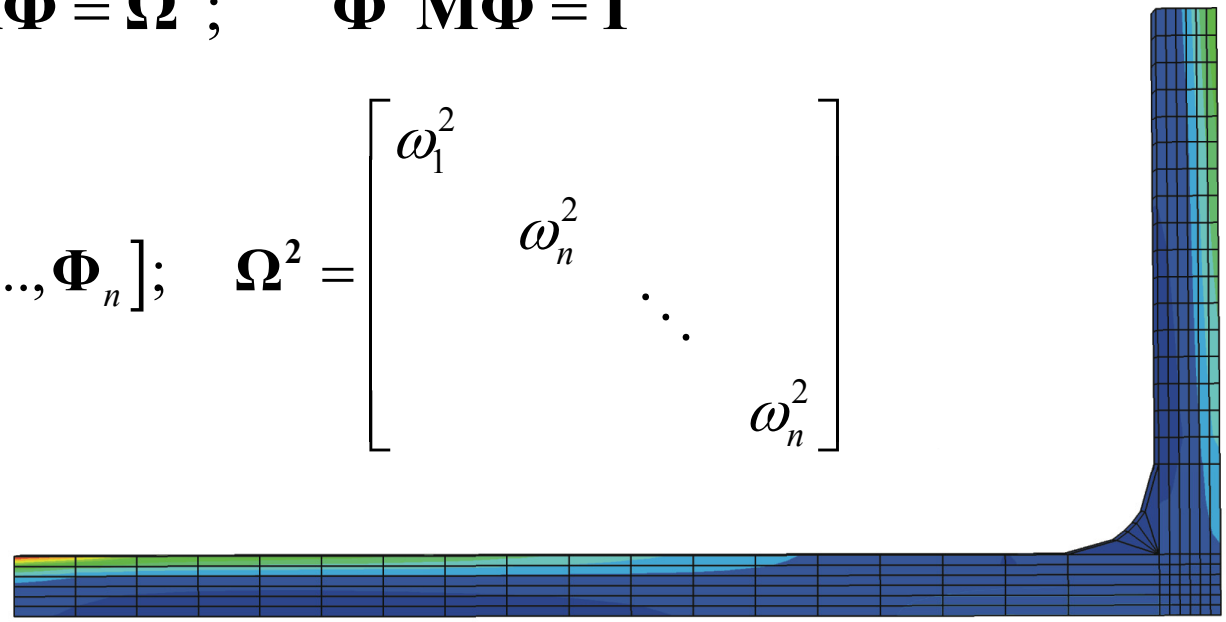
satisfy $\mathbf{M}\ddot{\mathbf{U}}+\mathbf{K}\mathbf{U}=0$

The *n* solutions may be written as:

$$\mathbf{K}\Phi=\mathbf{M}\Phi\Omega^2,\qquad \Phi^T\mathbf{K}\Phi=\Omega^2;\qquad \Phi^T\mathbf{M}\Phi=\mathbf{I}$$

with:

$$\Phi=\left[\Phi_1,\Phi_2,\ldots,\Phi_n\right];\quad \mathbf{\Omega^2}=\begin{bmatrix}\omega_1^2 & & & \\ & \omega_n^2 & & \\ & & \ddots & \\ & & & \omega_n^2\end{bmatrix}$$



The Objective - Mode Superposition

- **Change of Basis to Modal Coordinates**

Now using $\mathbf{U}(t) = \mathbf{\Phi}\mathbf{X}(t)$

in

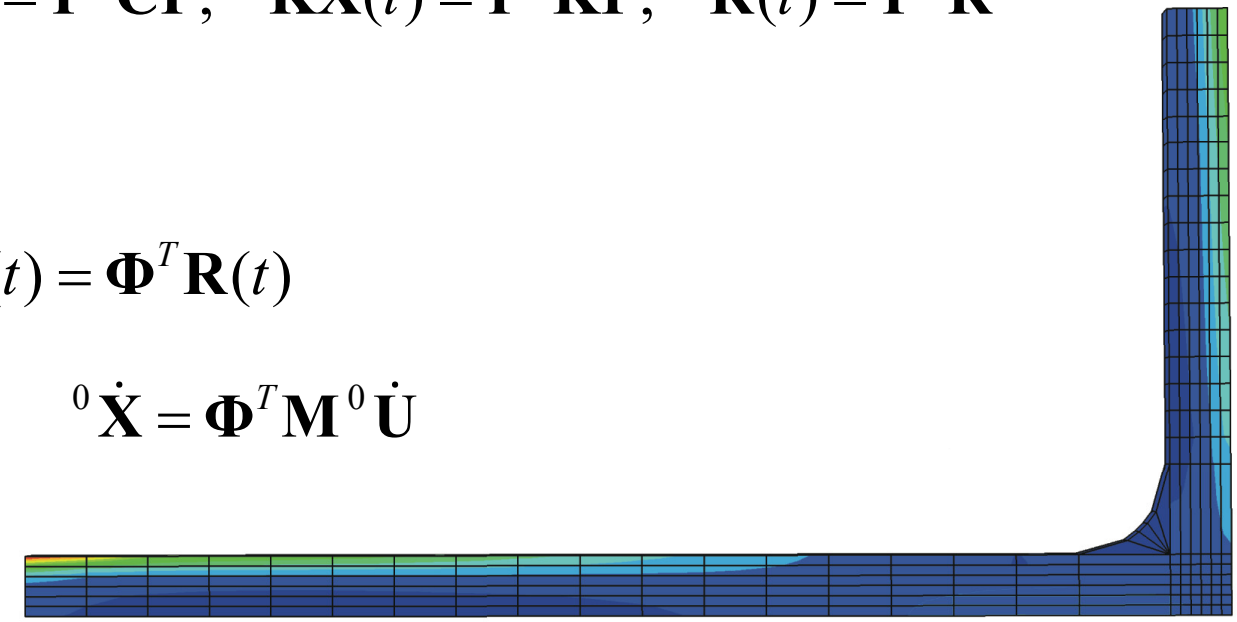
$$\tilde{\mathbf{M}}\ddot{\mathbf{X}}(t) + \tilde{\mathbf{C}}\dot{\mathbf{X}}(t) + \tilde{\mathbf{K}}\mathbf{X}(t) = \tilde{\mathbf{R}}(t)$$

$$\tilde{\mathbf{M}}\ddot{\mathbf{X}}(t) = \mathbf{P}^T\mathbf{M}\mathbf{P}, \quad \tilde{\mathbf{C}}\dot{\mathbf{X}}(t) = \mathbf{P}^T\mathbf{C}\mathbf{P}, \quad \tilde{\mathbf{K}}\mathbf{X}(t) = \mathbf{P}^T\mathbf{K}\mathbf{P}, \quad \tilde{\mathbf{R}}(t) = \mathbf{P}^T\mathbf{R}$$

we get

$$\ddot{\mathbf{X}}(t) + \mathbf{\Phi}^T\mathbf{C}\mathbf{\Phi}\dot{\mathbf{X}}(t) + \mathbf{\Omega}^2\mathbf{X}(t) = \mathbf{\Phi}^T\mathbf{R}(t)$$

with $^0\mathbf{X} = \mathbf{\Phi}^T\mathbf{M}\,^0\mathbf{U}; \qquad ^0\dot{\mathbf{X}} = \mathbf{\Phi}^T\mathbf{M}\,^0\dot{\mathbf{U}}$



Transformation Methods

- **Introduction**

We need to perform the transformations

$$\mathbf{\Phi}^T \mathbf{K} \mathbf{\Phi} = \mathbf{\Lambda}$$

$$\mathbf{\Phi}^T \mathbf{M} \mathbf{\Phi} = \mathbf{I}$$

The transformation may be pursued by iteration

$$\mathbf{K}_2 = \mathbf{P}_1^T \mathbf{K}_1 \mathbf{P}_1 \qquad \mathbf{M}_2 = \mathbf{P}_1^T \mathbf{M}_1 \mathbf{P}_1$$

$$\mathbf{K}_3 = \mathbf{P}_2^T \mathbf{K}_2 \mathbf{P}_2 \qquad \mathbf{M}_3 = \mathbf{P}_2^T \mathbf{M}_2 \mathbf{P}_2$$

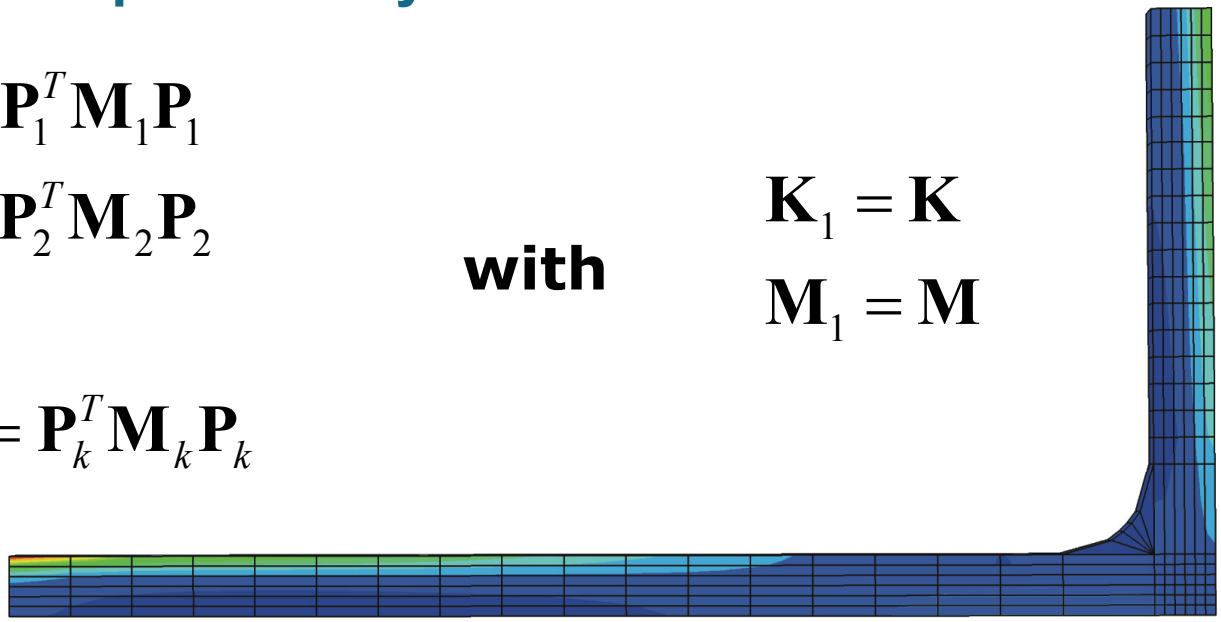
$$\vdots \qquad\qquad \vdots$$

$$\mathbf{K}_{k+1} = \mathbf{P}_k^T \mathbf{K}_k \mathbf{P}_k \qquad \mathbf{M}_{k+1} = \mathbf{P}_k^T \mathbf{M}_k \mathbf{P}_k$$

with

$$\mathbf{K}_1 = \mathbf{K}$$

$$\mathbf{M}_1 = \mathbf{M}$$



Transformation Methods

- **Introduction**

The aim being to select $\mathbf{P}_k$ such as to bring $\mathbf{K}_k, \mathbf{M}_k$ closer to diagonal form i.e. :

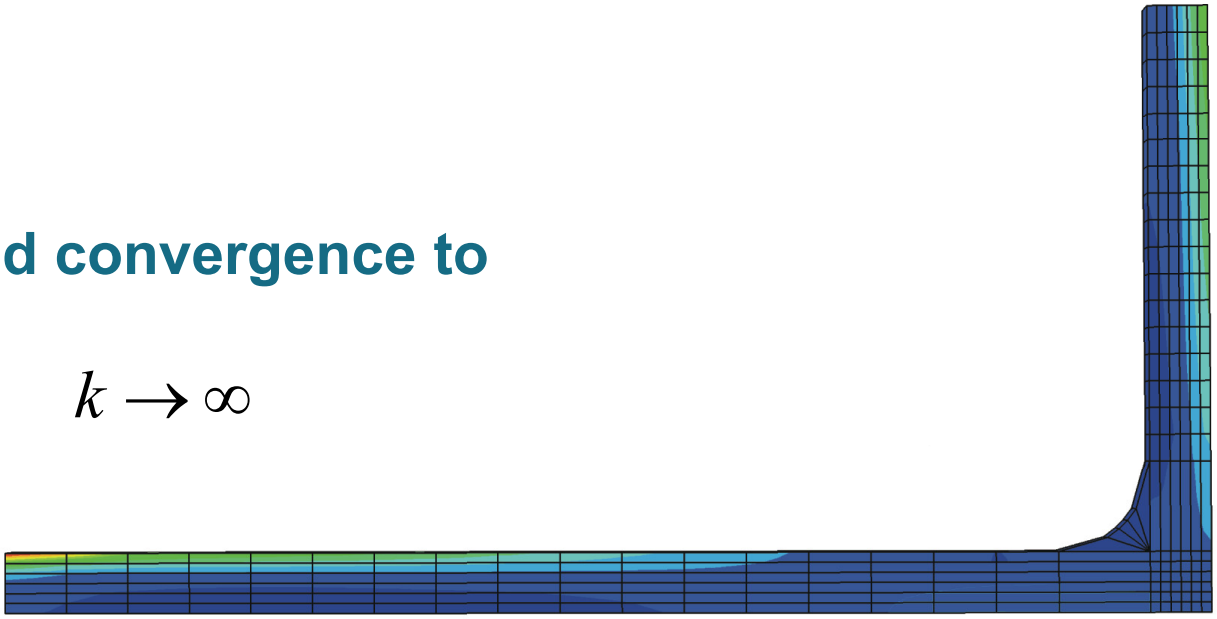
$$\mathbf{K}_{k+1} \to \mathbf{\Lambda}, \quad \mathbf{M}_{k+1} \to \mathbf{I}, \quad k \to \infty$$

whereby there is:

$$\mathbf{\Phi} = \mathbf{P}_1 \cdot \mathbf{P}_2 \ldots \mathbf{P}_l$$

in practice we don't need convergence to

$$\mathbf{K}_{k+1} \to \mathbf{\Lambda}, \quad \mathbf{M}_{k+1} \to \mathbf{I}, \quad k \to \infty$$



Transformation Methods

- **Introduction**

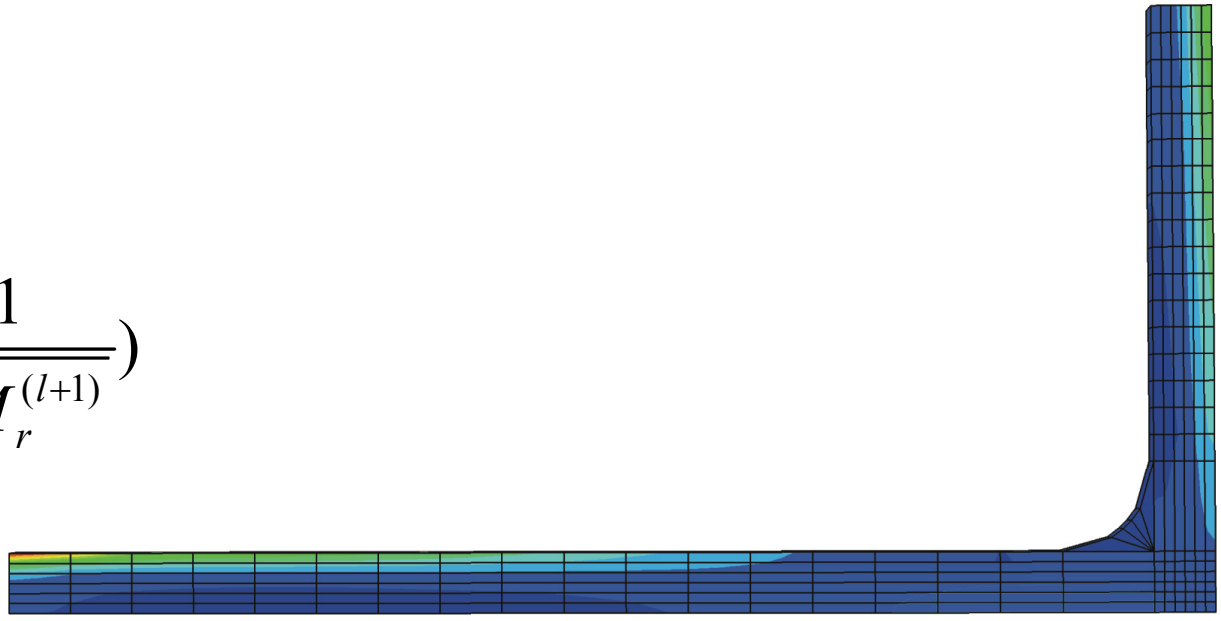
In practice we don't need convergence to $\mathbf{\Lambda}$, $\mathbf{I}$ only to diagonal form;

$$\mathbf{K}_{k+1} \to diag(K_r), \quad \mathbf{M}_{k+1} \to diag(M_r), \quad k \to \infty$$

If *l* is the last iteration there is:

$$\mathbf{\Lambda} = diag\left(\frac{K_r^{(l+1)}}{M_r^{(l+1)}}\right)$$

$$\mathbf{\Phi} = \mathbf{P_1} \cdot \mathbf{P_2} \ldots \mathbf{P}_l \; diag\left(\frac{1}{\sqrt{M_r^{(l+1)}}}\right)$$



Transformation Methods

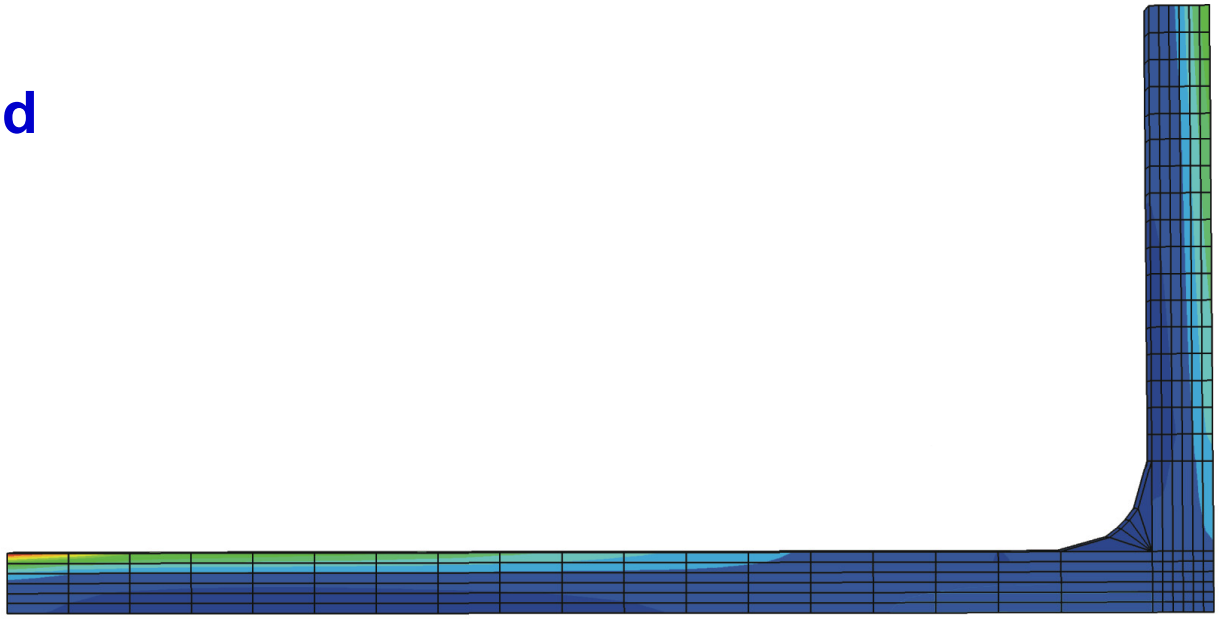
- **Introduction**

  Based on these consideration a number of iteration schemes have been proposed here we will consider the

  Jacobi

  and the

  Householder-QR-method



Transformation Methods

- **The Jacobi Method (M = I)**

This method can be applied to calculate negative, zero and positive eigenvalues!

$$\mathbf{K}\Phi = \lambda\Phi$$

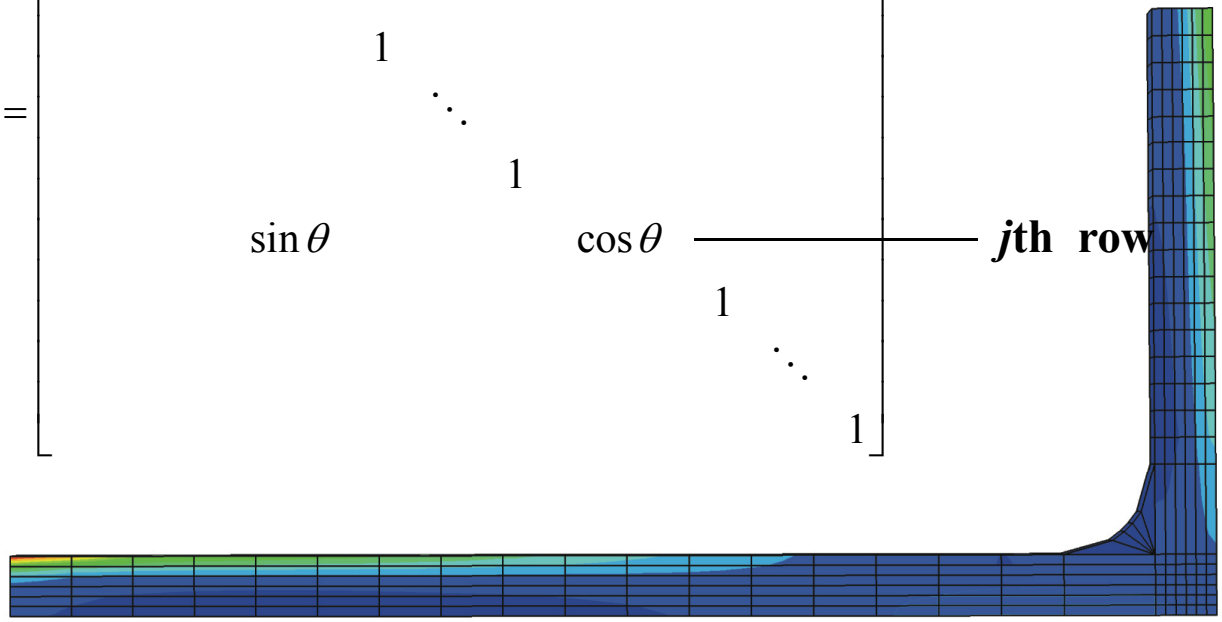
$$\Downarrow$$

$$\mathbf{K}_{k+1} = \mathbf{P}_k^T \mathbf{K}_k \mathbf{P}^T$$

where θ is selected such that element (*i*,*j*) in K_{k+1} becomes zero

$$\mathbf{P}_k = \begin{bmatrix} 1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & \cos\theta & & & -\sin\theta & & \\ & & & & 1 & & & & \\ & & & & & \ddots & & & \\ & & & & & & 1 & & \\ & & & \sin\theta & & & \cos\theta & & \\ & & & & & & & 1 & \\ & & & & & & & & \ddots \\ & & & & & & & & & 1 \end{bmatrix}$$

*i*th — *j*th column — *i*th row — *j*th row



Transformation Methods

$$\mathbf{K}_{k+1} = \mathbf{P}_k^T \mathbf{K}_k \mathbf{P}^T$$

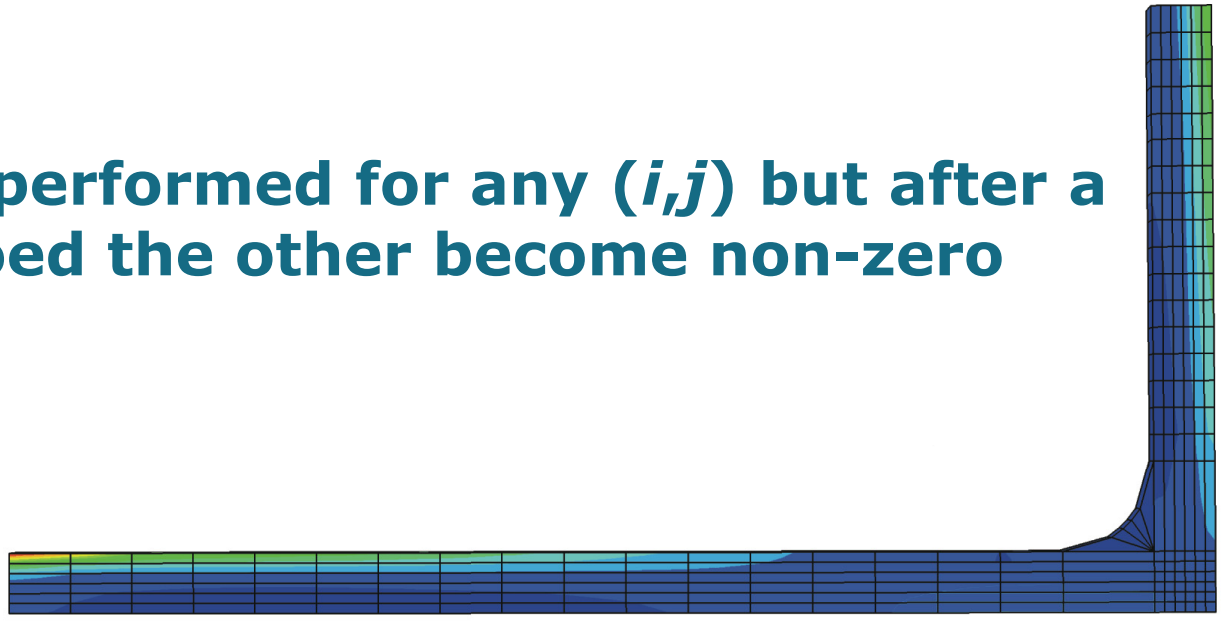
- **The Jacobi Method**

where θ is selected such that element (*i,j*) in K_{k+1} becomes zero, i.e.:

$$\tan 2\theta = \frac{2k_{ij}^{(k)}}{k_{ii}^{(k)} - k_{jj}^{(k)}} \quad \textbf{for} \quad k_{ii}^{(k)} \neq k_{jj}^{(k)}$$

$$\theta = \frac{\theta}{4} \quad \textbf{for} \quad k_{ii}^{(k)} = k_{jj}^{(k)}$$

this zeroing can be performed for any (*i,j*) but after a new element is zeroed the other become non-zero again



Transformation Methods

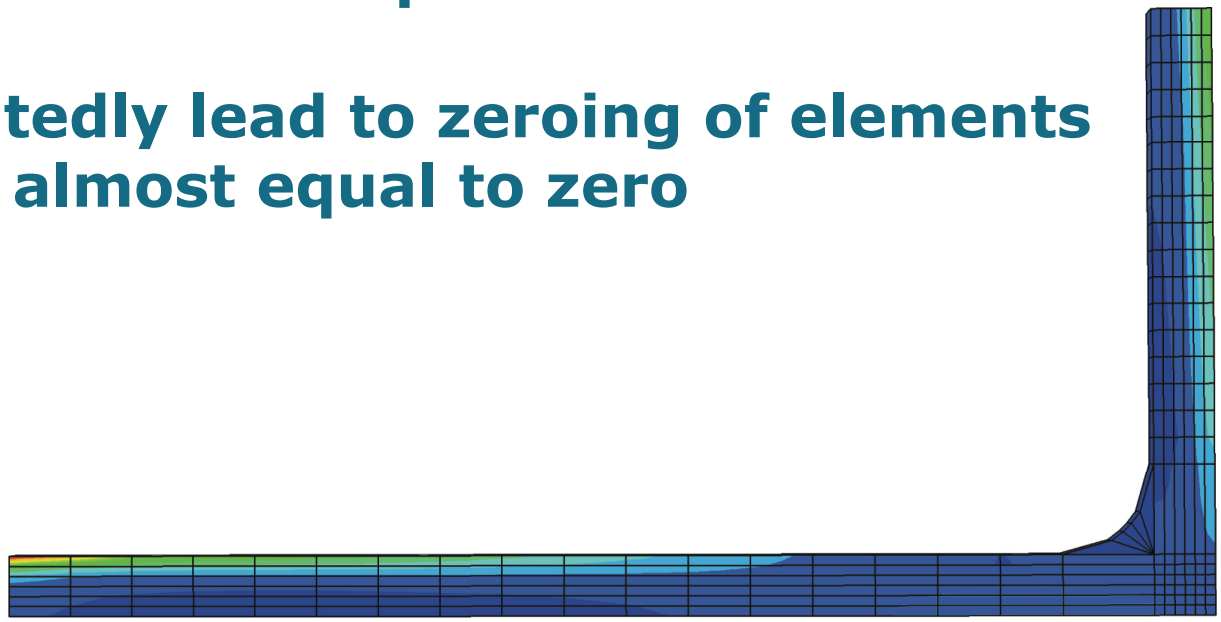
$$\mathbf{K}_{k+1} = \mathbf{P}_k^T \mathbf{K}_k \mathbf{P}^T$$

- **The Jacobi Method**

  A strategy is therefore to always zero the off diagonal elements furthest away from the diagonal – but this is time consuming (the searching process)

  Another approach is simply to systematically go through all elements - sweep after sweep

  – but this will repeatedly lead to zeroing of elements which are already almost equal to zero



Transformation Methods

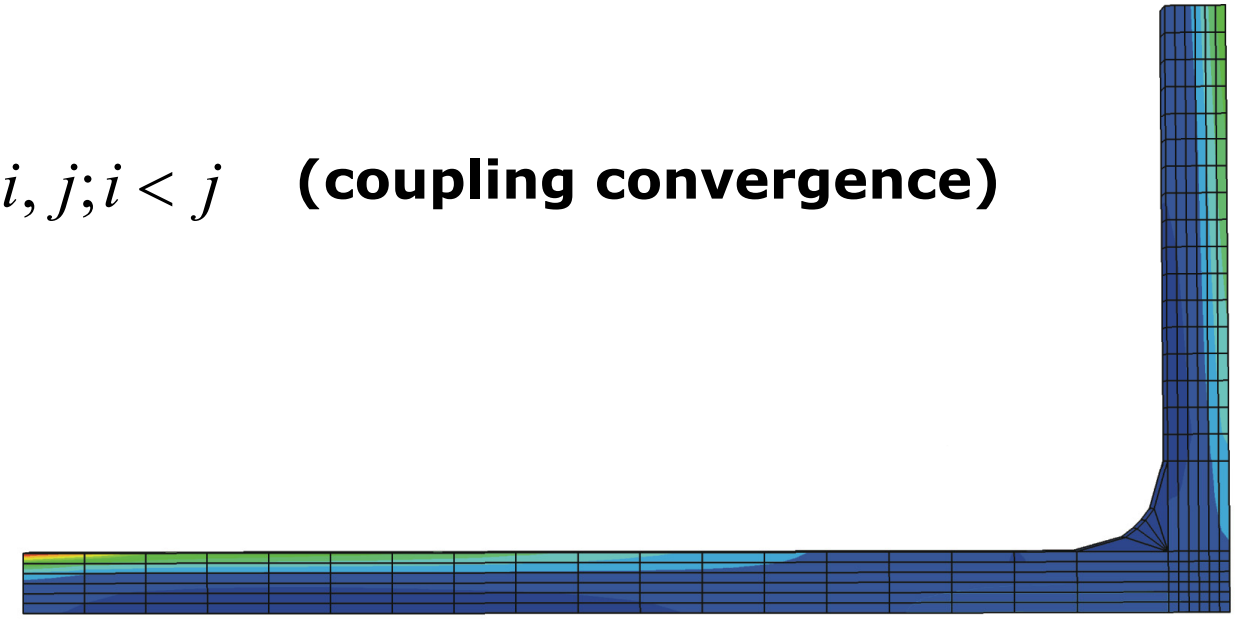
$$\mathbf{K}_{k+1} = \mathbf{P}_k^T \mathbf{K}_k \mathbf{P}^T$$

- **The Jacobi Method**

A threshold Jacobi method may be formulated such that only elements larger than a certain threshold are zeroed i.e. convergence is achieved when:

$$\frac{k_{ii}^{(l+1)} - k_{ii}^{(l)}}{k_{ii}^{(l+1)}} \leq 10^{-s} \qquad i = 1, 2..., n \qquad \textbf{(zero convergence)}$$

$$\frac{\left(k_{ij}^{(l+1)}\right)^2}{k_{ii}^{(l+1)} k_{jj}^{(l+1)}} \leq 10^{-s} \qquad \textbf{all} \quad i, j; i < j \qquad \textbf{(coupling convergence)}$$



Transformation Methods

$$\mathbf{K}_{k+1} = \mathbf{P}_k^T \mathbf{K}_k \mathbf{P}^T$$

- **The Jacobi Method**

  The procedure is summarized as:

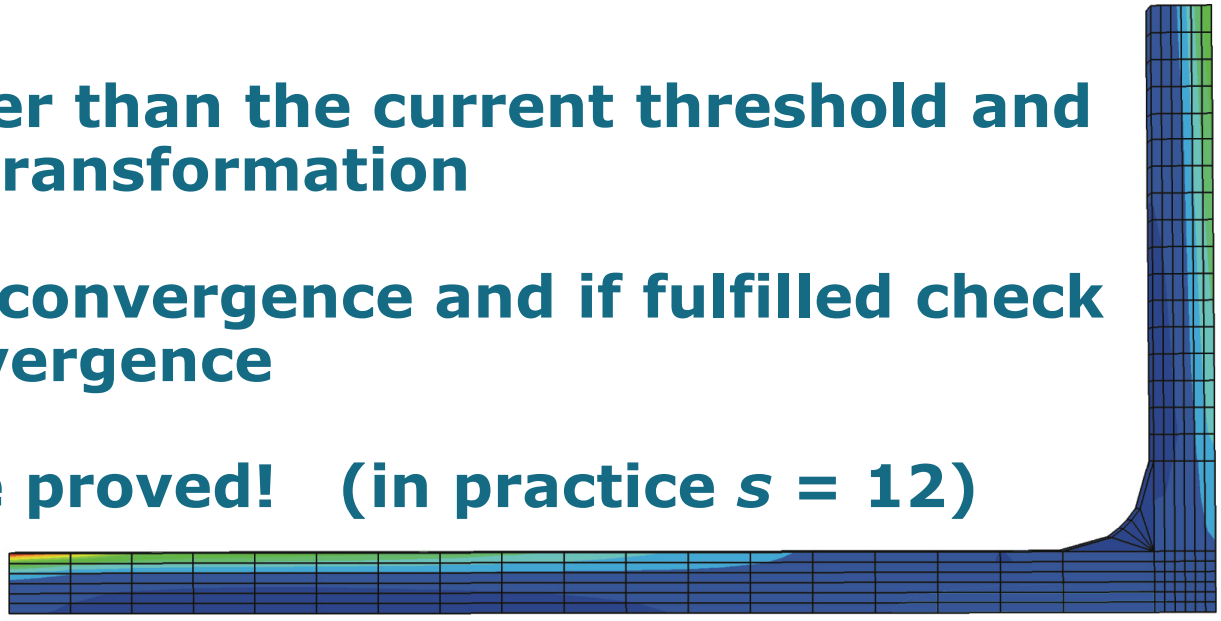
  1) Initialize the threshold for the *m*th sweep (10^{-2m})
  2) For all *i,j* with *i<j* calculate the coupling factor

$$\frac{\left(k_{ij}^{(l+1)}\right)^2}{k_{ii}^{(l+1)} k_{jj}^{(l+1)}}$$

  check if it is larger than the current threshold and only then apply transformation

  3) Check for (zero) convergence and if fulfilled check for coupling convergence

  Convergence can be proved! (in practice *s* = 12)



Transformation Methods

- **The Generalized Jacobi Method (M ≠ I)**
  Operates directly on K and M

$$\mathbf{K\Phi} = \lambda \mathbf{M\Phi}$$

α and γ are selected such as to diagonalize K and M simultaneously

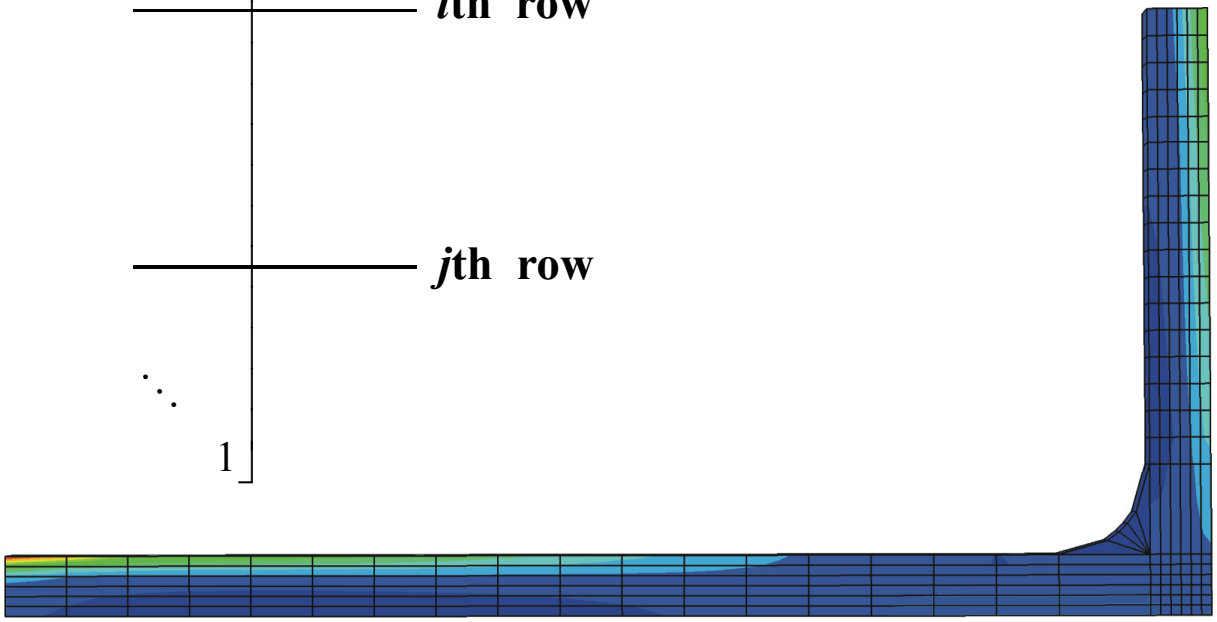
$$\mathbf{P}_k = \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & \alpha & & \\ & & & \ddots & & & \\ & & \gamma & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}$$

*i*th column

*j*th column

*i*th row

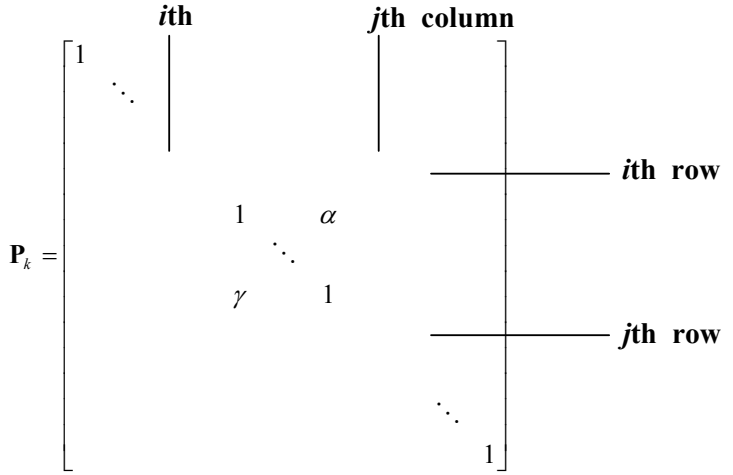
*j*th row



Transformation Methods

- **The Generalized Jacobi Method (M ≠ I)**

$$\mathbf{K\Phi} = \boldsymbol{\lambda}\mathbf{M\Phi}$$



Performing the multiplications

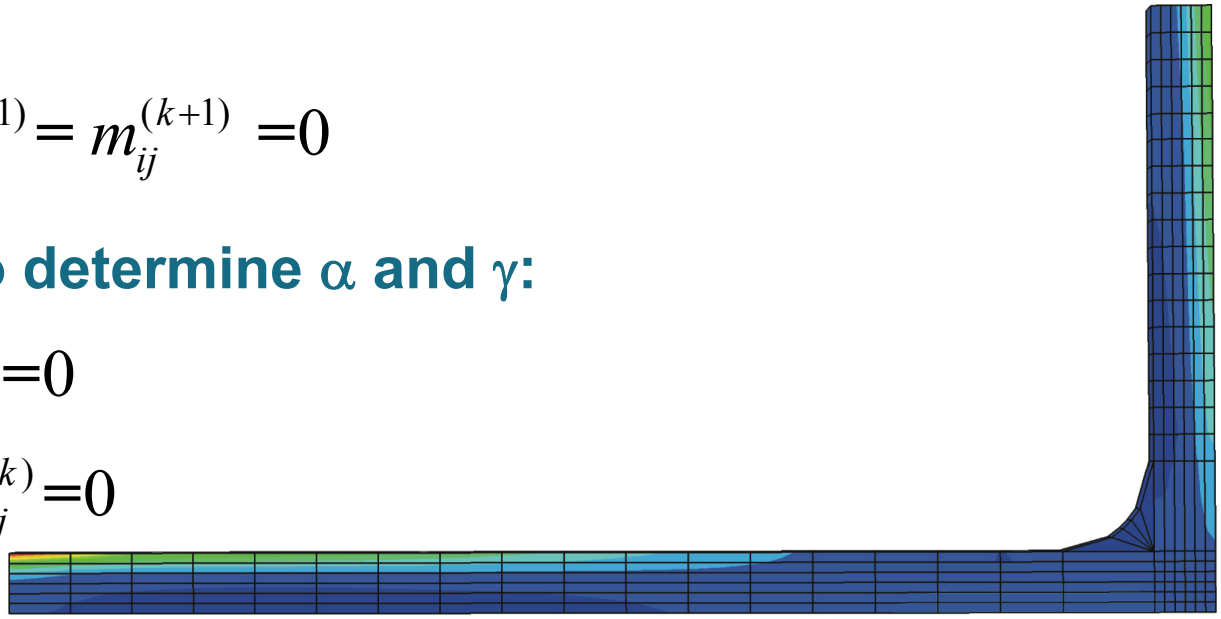
$$\mathbf{P}_k^T\mathbf{K}_k\mathbf{P}_k \qquad \mathbf{P}_k^T\mathbf{M}_k\mathbf{P}_k$$

and requiring that : $k_{ij}^{(k+1)} = m_{ij}^{(k+1)} = 0$

we get two equations to determine α and γ:

$$\alpha k_{ii}^{(k)} + (1+\alpha\gamma)k_{ij}^{(k)} + \gamma k_{jj}^{(k)} = 0$$

$$\alpha m_{ii}^{(k)} + (1+\alpha\gamma)m_{ij}^{(k)} + \gamma m_{jj}^{(k)} = 0$$



Transformation Methods

$$\alpha k_{ii}^{(k)} + (1+\alpha\gamma) k_{ij}^{(k)} + \gamma k_{jj}^{(k)} = 0$$
$$\alpha m_{ii}^{(k)} + (1+\alpha\gamma) m_{ij}^{(k)} + \gamma m_{jj}^{(k)} = 0$$

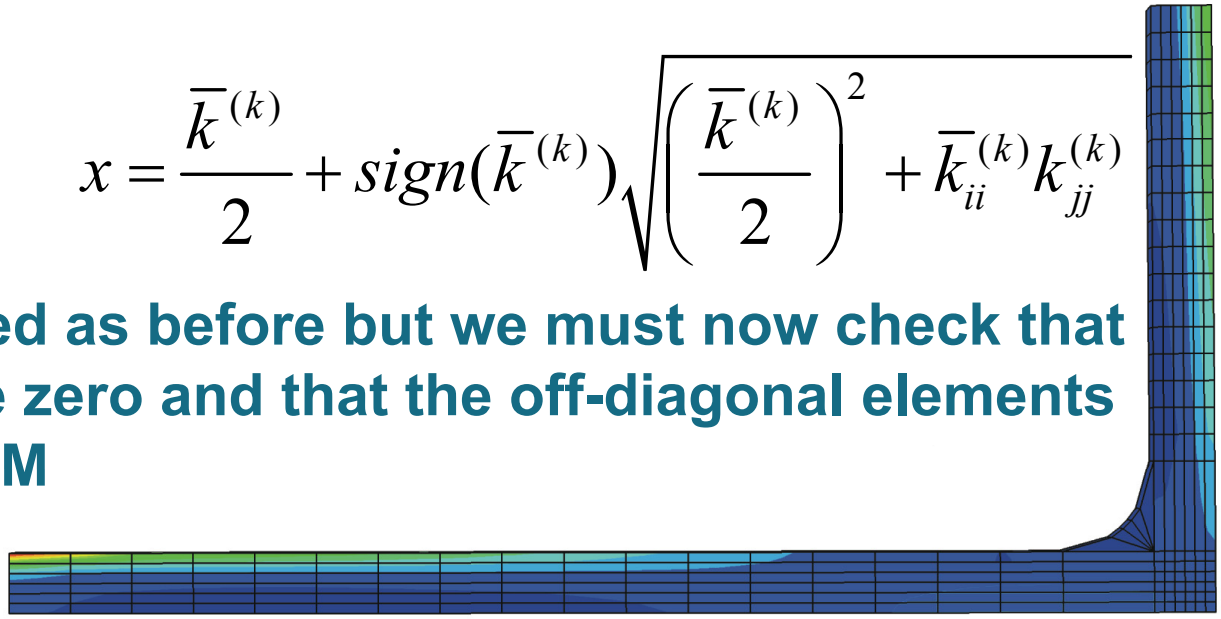
- **The Generalized Jacobi Method (M ≠ I)**

We may solve the two equations from:

$$\bar{k}_{ii}^{(k)} = k_{ii}^{(k)} m_{ij}^{(k)} - m_{ii}^{(k)} k_{ij}^{(k)}$$
$$\bar{k}_{jj}^{(k)} = k_{jj}^{(k)} m_{ij}^{(k)} - m_{jj}^{(k)} k_{ij}^{(k)}$$
$$\bar{k}^{(k)} = k_{ii}^{(k)} m_{jj}^{(k)} - k_{ii}^{(k)} m_{ii}^{(k)}$$

$$\gamma = -\frac{\bar{k}_{ii}^{(k)}}{x}; \qquad \alpha = \frac{\bar{k}_{jj}^{(k)}}{x}, \qquad x = \frac{\bar{k}^{(k)}}{2} + sign(\bar{k}^{(k)}) \sqrt{\left(\frac{\bar{k}^{(k)}}{2}\right)^2 + \bar{k}_{ii}^{(k)} k_{jj}^{(k)}}$$

the iteration is performed as before but we must now check that the coupling factors are zero and that the off-diagonal elements are zero for both K and M



Transformation Methods

- **The Householder-QR-Inverse Iteration Solution**

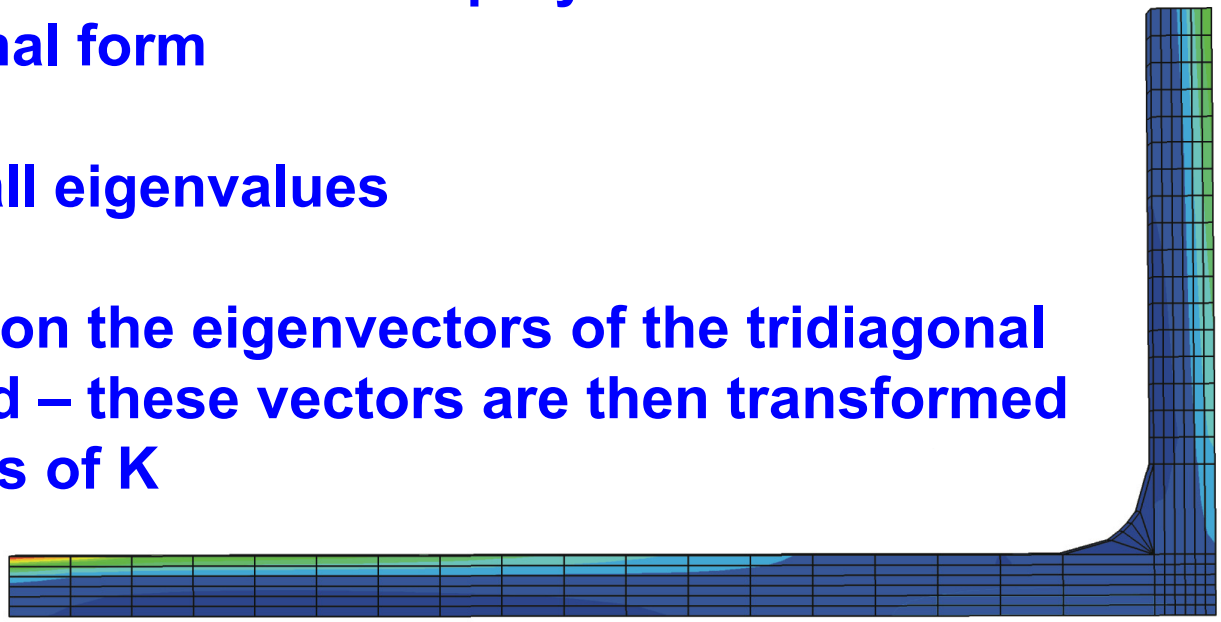
  We consider here $\mathbf{K\Phi = \lambda\Phi}$

  The HQRI solution method stands for the following steps:

  1) **Householder transformations are employed to reduce the matrix K to tridiagonal form**

  2) **QR iteration yields all eigenvalues**

  3) **Using inverse iteration the eigenvectors of the tridiagonal matrix are calculated – these vectors are then transformed into the eigenvectors of K**



Transformation Methods

- **The Householder-QR-Inverse Iteration Solution**

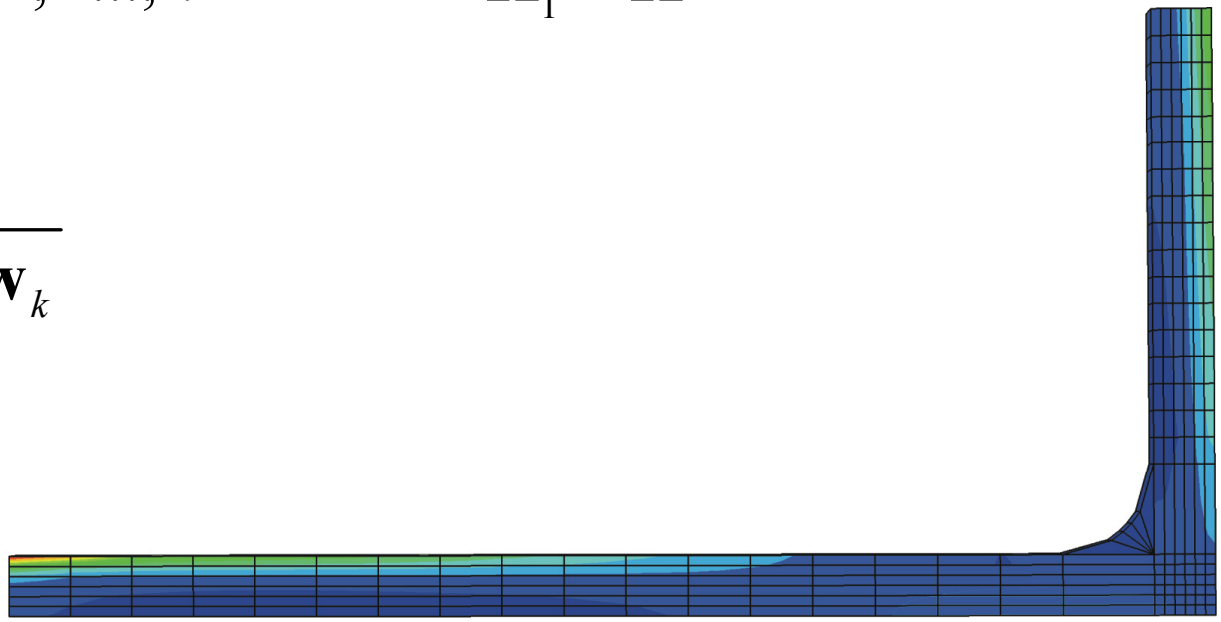
The Householder transformations:

We transform K to tridiagonal form by *n*-2 transformations

$$\mathbf{K}_{k+1} = \mathbf{P}_k^T \mathbf{K}_k \mathbf{P}_k, \qquad k = 1, 2..., n-2 \qquad \mathbf{K}_1 = \mathbf{K}$$

$$\mathbf{P}_k = \mathbf{I} - \theta \mathbf{w}_k \mathbf{w}_k^T, \quad \theta = \frac{2}{\mathbf{w}_k^T \mathbf{w}_k}$$

Reflection matrix



Transformation Methods

- **The Householder-QR-Inverse Iteration Solution**

The Householder transformations:

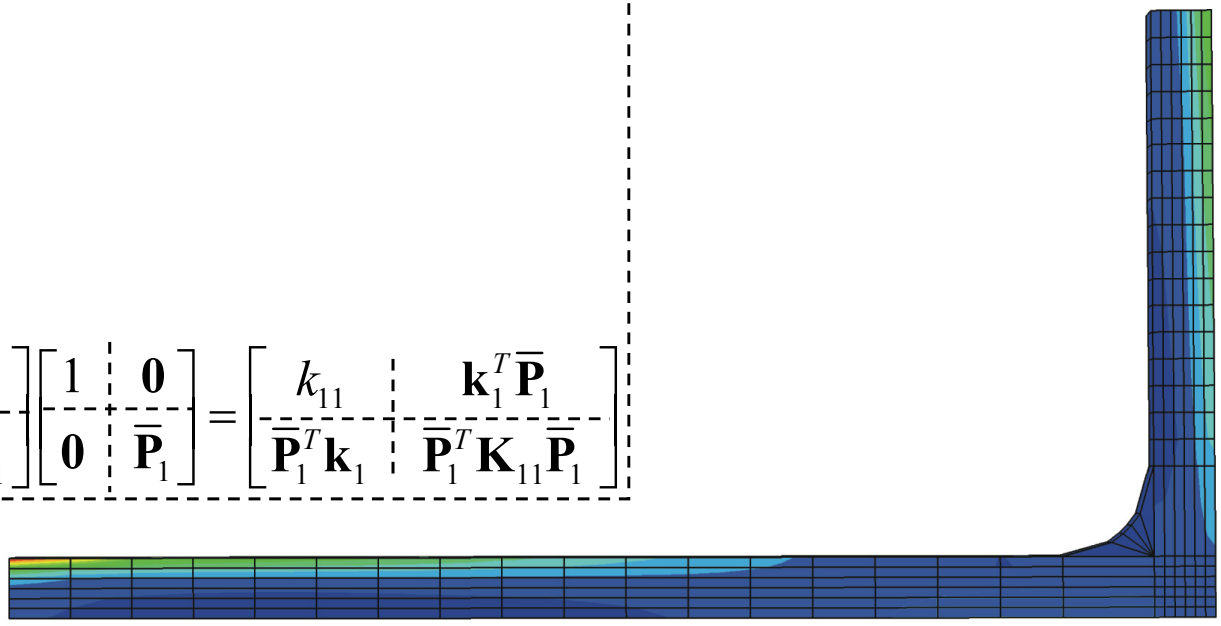
We may determine w_k from:

$$\mathbf{P}_1 = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{P}}_1 \end{bmatrix}, w_1 = \begin{bmatrix} 0 \\ \bar{w}_1 \end{bmatrix}$$

$$\mathbf{K}_1 = \begin{bmatrix} k_{11} & \mathbf{k}_1^T \\ \mathbf{k}_1 & \mathbf{K}_{11} \end{bmatrix}$$

$$\Downarrow$$

$$\mathbf{K}_2 = \mathbf{P}_1^T \mathbf{K}_1 \mathbf{P}_1 = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{P}}_1 \end{bmatrix} \begin{bmatrix} k_{11} & \mathbf{k}_1^T \\ \mathbf{k}_1 & \mathbf{K}_{11} \end{bmatrix} \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{P}}_1 \end{bmatrix} = \begin{bmatrix} k_{11} & \mathbf{k}_1^T \bar{\mathbf{P}}_1 \\ \bar{\mathbf{P}}_1^T \mathbf{k}_1 & \bar{\mathbf{P}}_1^T \mathbf{K}_{11} \bar{\mathbf{P}}_1 \end{bmatrix}$$



Transformation Methods

- **The Householder-QR-Inverse Iteration Solution**

The Householder transformations:

$$\mathbf{P}_1 = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{P}}_1 \end{bmatrix}, w_1 = \begin{bmatrix} 0 \\ \bar{w}_1 \end{bmatrix}$$

$$\mathbf{K}_1 = \begin{bmatrix} k_{11} & \mathbf{k}_1^T \\ \mathbf{k}_1 & \mathbf{K}_{11} \end{bmatrix}$$

$$\Downarrow$$

$$\mathbf{K}_2 = \mathbf{P}_1^T \mathbf{K}_1 \mathbf{P}_1 = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{P}}_1 \end{bmatrix} \begin{bmatrix} k_{11} & \mathbf{k}_1^T \\ \mathbf{k}_1 & \mathbf{K}_{11} \end{bmatrix} \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{P}}_1 \end{bmatrix} = \begin{bmatrix} k_{11} & \mathbf{k}_1^T \bar{\mathbf{P}}_1 \\ \bar{\mathbf{P}}_1^T \mathbf{k}_1 & \bar{\mathbf{P}}_1^T \mathbf{K}_{11} \bar{\mathbf{P}}_1 \end{bmatrix}$$

We would like K_2 in the form:

$$\mathbf{K}_2 = \begin{bmatrix} k_{11} & \times & 0 & \dots & 0 \\ \times & & & & \\ 0 & & & \bar{\mathbf{K}}_2 & \\ \vdots & & & & \\ 0 & & & & \end{bmatrix}$$

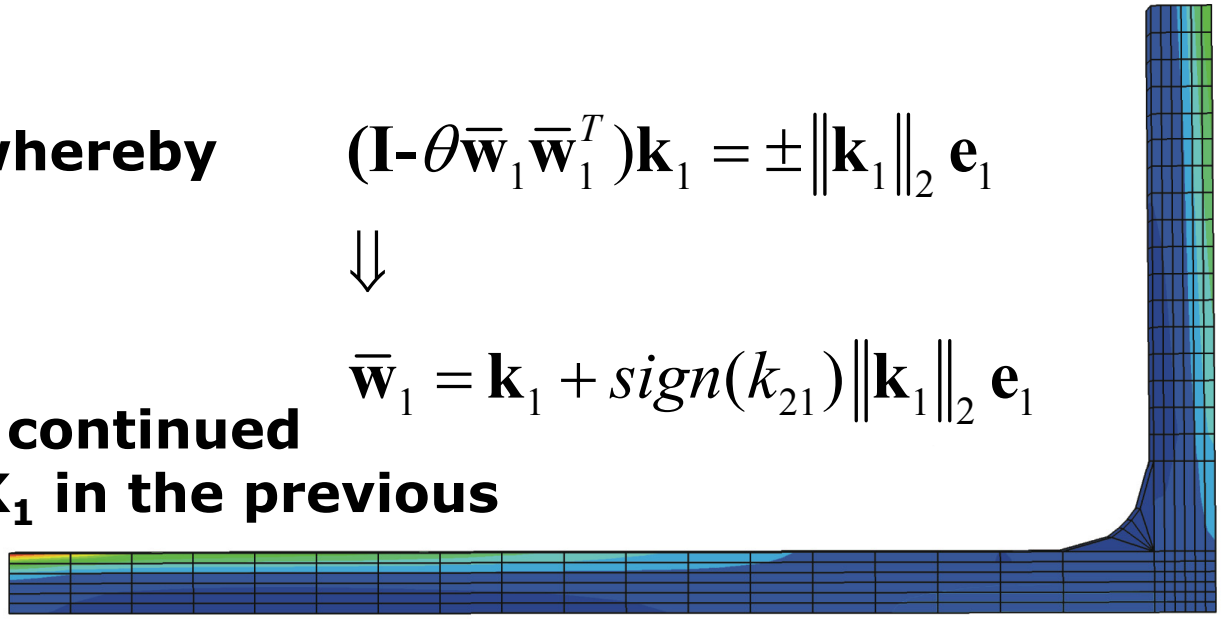
whereby

$$(\mathbf{I} - \theta \bar{\mathbf{w}}_1 \bar{\mathbf{w}}_1^T)\mathbf{k}_1 = \pm \|\mathbf{k}_1\|_2 \mathbf{e}_1$$

$$\Downarrow$$

$$\bar{\mathbf{w}}_1 = \mathbf{k}_1 + sign(k_{21}) \|\mathbf{k}_1\|_2 \mathbf{e}_1$$

The procedure is now continued by considering K_2 as K_1 in the previous



Transformation Methods

- **The Householder-QR-Inverse Iteration Solution**

  The QR iteration

  The basic step is to decompose K into the form:

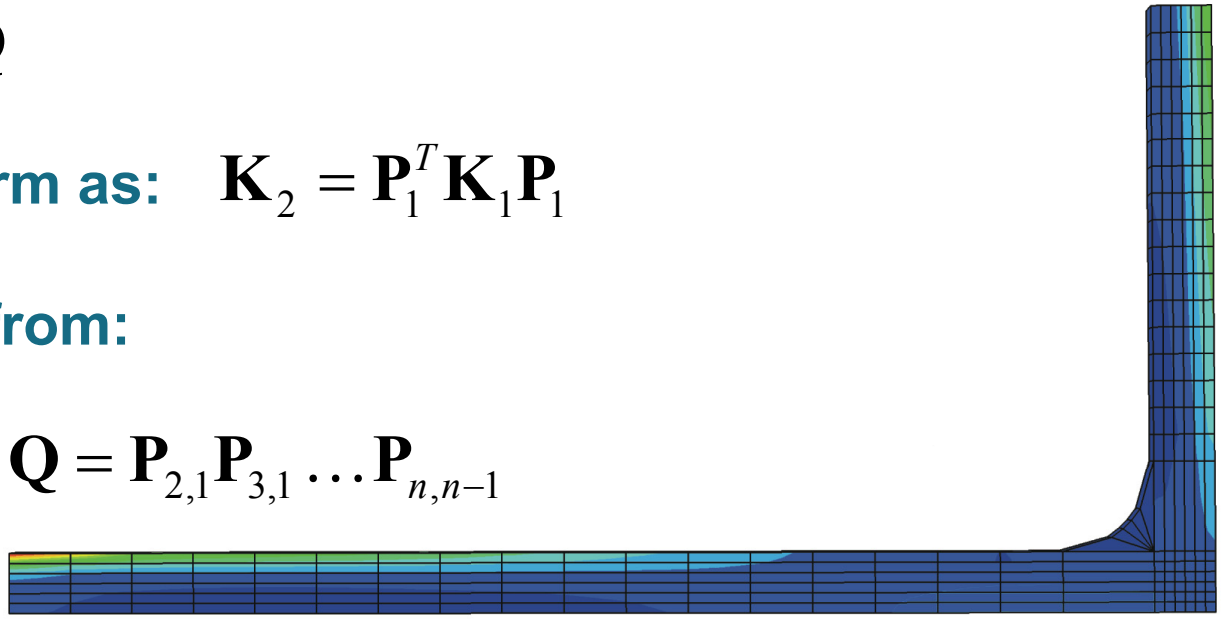
$$\mathbf{K} = \mathbf{QR}$$

  and then $\mathbf{RQ} = \mathbf{Q}^T\mathbf{KQ}$

  which is of the same form as: $\mathbf{K}_2 = \mathbf{P}_1^T\mathbf{K}_1\mathbf{P}_1$

  R and Q may be found from:

$$\mathbf{P}_{n,n-1}^T \ldots \mathbf{P}_{3,1}^T\mathbf{P}_{2,1}^T\mathbf{K} = \mathbf{R} \Rightarrow \mathbf{Q} = \mathbf{P}_{2,1}\mathbf{P}_{3,1} \ldots \mathbf{P}_{n,n-1}$$



Transformation Methods

- **The Householder-QR-Inverse Iteration Solution**

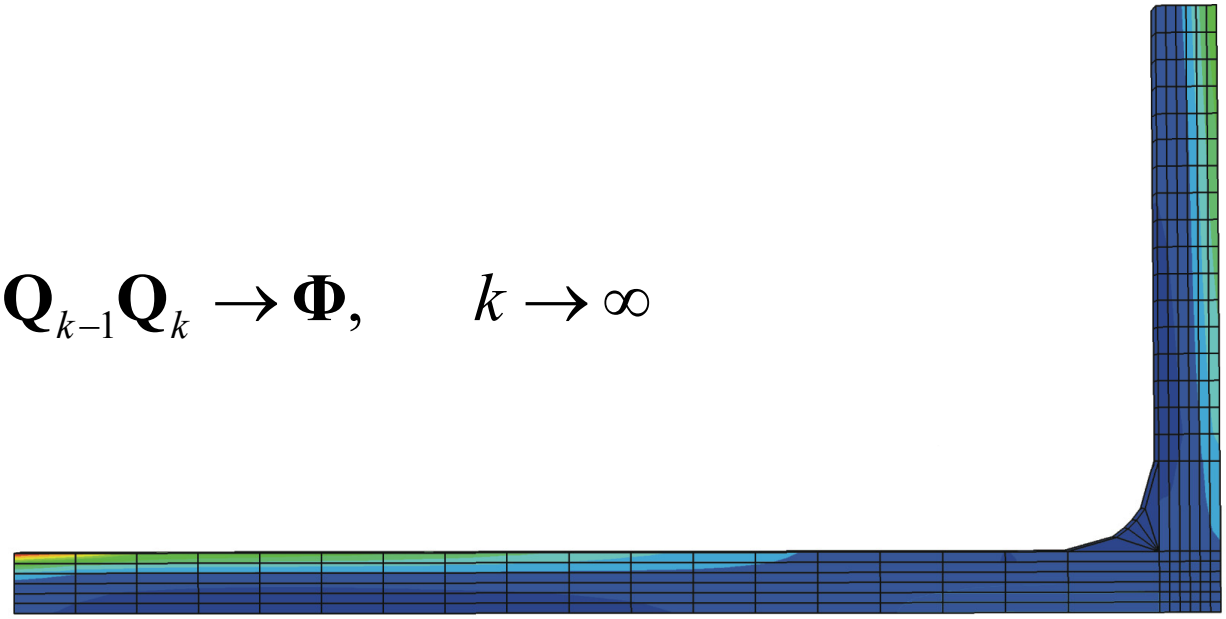
The QR iteration

We may now iterate:

$$\mathbf{K}_k = \mathbf{Q}_k \mathbf{R}_k$$

$$\mathbf{K}_{k+1} = \mathbf{R}_k \mathbf{Q}_k$$

$$\mathbf{K}_{k+1} \to \mathbf{\Lambda} \quad \textbf{and} \quad \mathbf{Q}_1 \ldots \mathbf{Q}_{k-1} \mathbf{Q}_k \to \mathbf{\Phi}, \qquad k \to \infty$$



Transformation Methods

- **The Householder-QR-Inverse Iteration Solution**

  The QR iteration

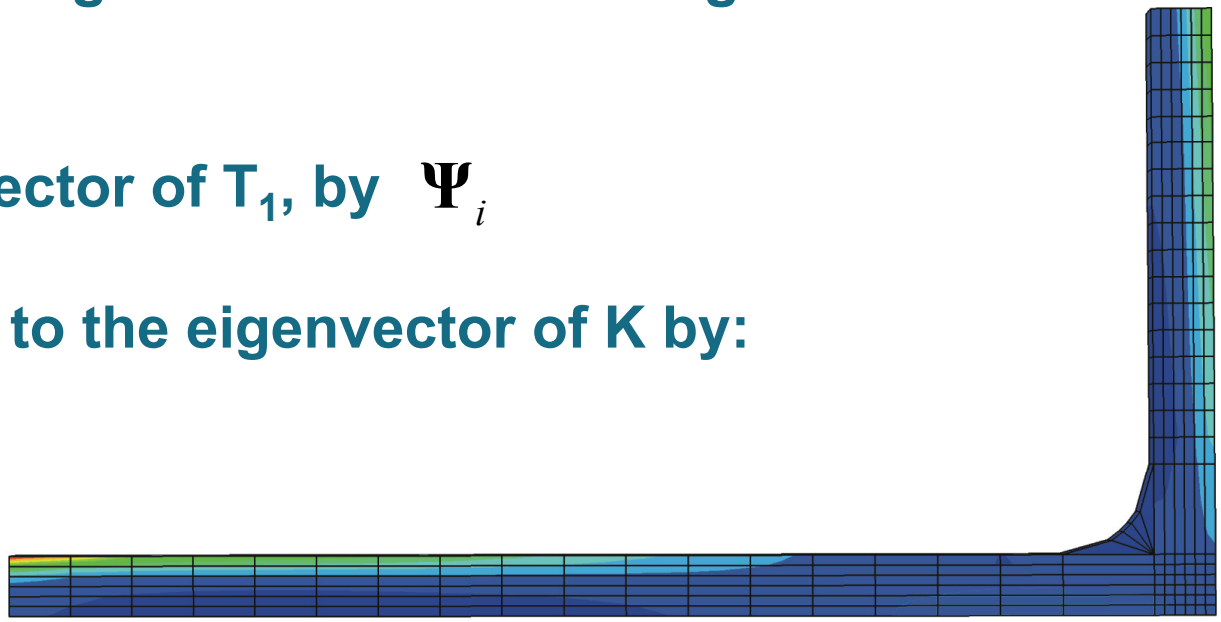
  As we started out with the QR method should have been used on the tridiagonalized K matrix

  If we do that we find the eigenvectors of the triangularized K matrix i.e. K_{n-1} or T_1

  Denoting the *i*th eigenvector of T_1, by Ψ_i

  we may transform back to the eigenvector of K by:

$$\Phi_i = \mathbf{P}_1 \mathbf{P}_2 \ldots \mathbf{P}_{n-2} \Psi_i$$



Very Condensed Summary

- **Non-Linear Finite Element Method**

  Basic principle – we sub-divide the loading in time steps, linearize the equlibrium equations (tankent stiffness matrix) and iterate for the solution (Newton Rapson/Modified NR, quasi Newton)

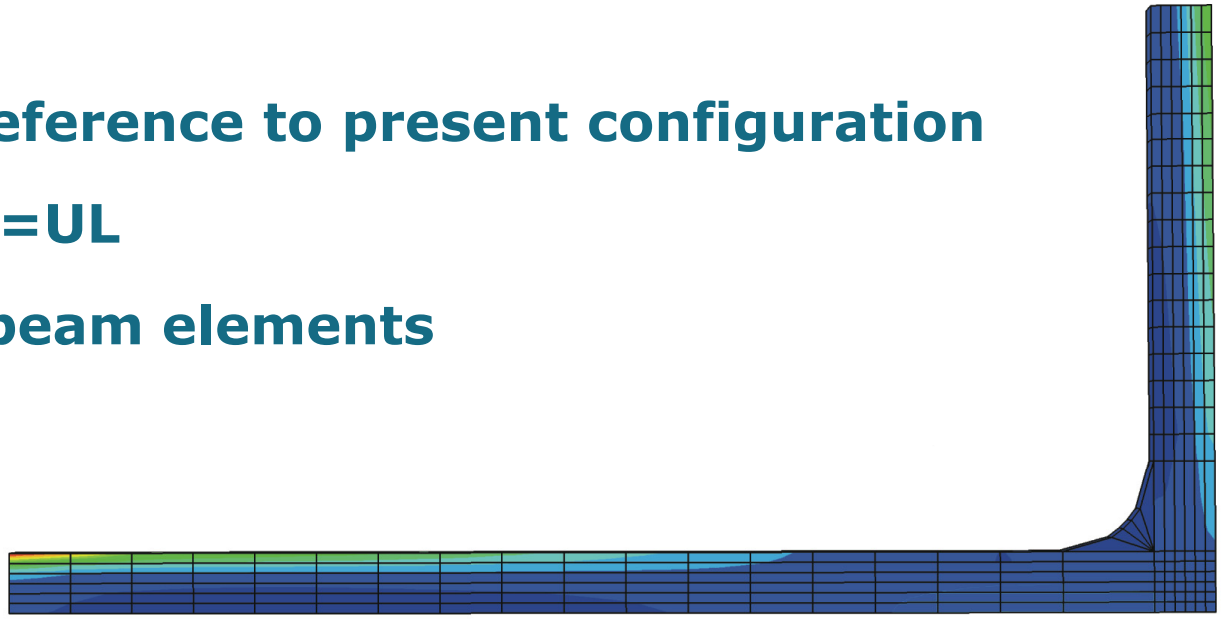
  Cauchy stress tensor, Green-Lagrange strain tensor, Almansi strain tensor, second Piola Kirchoff stress tensor

  Total Lagrange formulation – reference to original configuration

  Updated Lagrange – reference to present configuration

  Material non-linear TL=UL

  Analysis of truss and beam elements

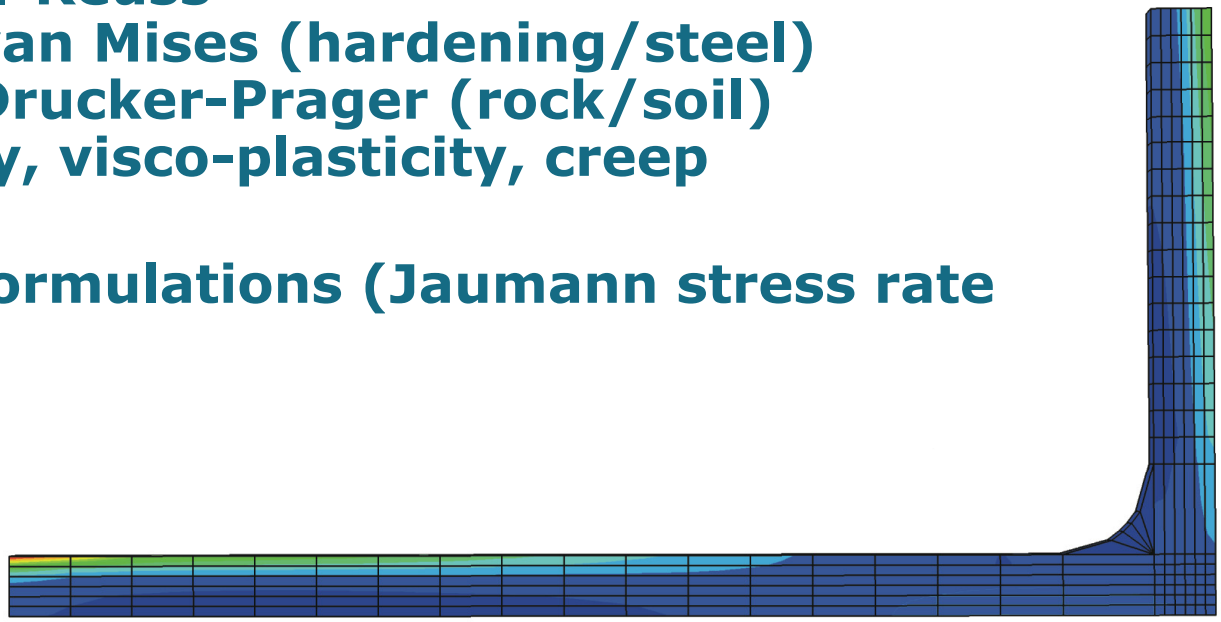


Very Condensed Summary

- **Non-Linear Finite Element Method**

  ISO-parametric elements
  2D: Axisymmetric element, plane strain, plane stress
  3D: Solid elements, beam and axi-symmetric shell elements

  Constitutive relations for non-linear materials
  Elasto-plastic Prandtl-Reuss
  van Mises (hardening/steel)
  Drucker-Prager (rock/soil)
  Thermoelastoplasticity, visco-plasticity, creep
  Elasto-plasticity
  Rate based formulations (Jaumann stress rate velocity)
  Contact problems



Very Condensed Summary

- **Dynamic Finite Element Method**

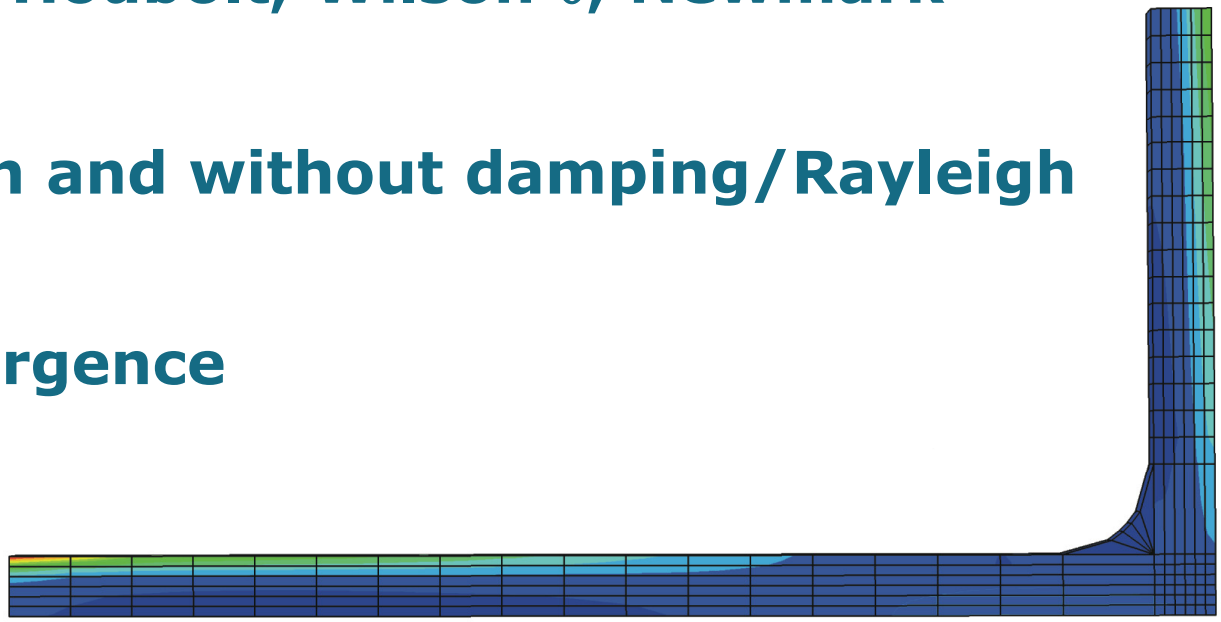
  Direct integration – subdivide tiem into steps, assume a certain variation of the motion - and integrate

  Explicit – reference to t (no factorization)
  Imlicit – reference to t+Δt (factorization)

  Central differences, Houbolt, Wilson θ, Newmark

  Modal analysis (with and without damping/Rayleigh damping)

  Precision and convergence



Very Condensed Summary

- **Dynamic Finite Element Method**

  Material non-linear dynamics – iterate within each time step based on a linearization of the stiffness matrix

  Eigenvalue problems

  - vector iteration method
  - transformation methods