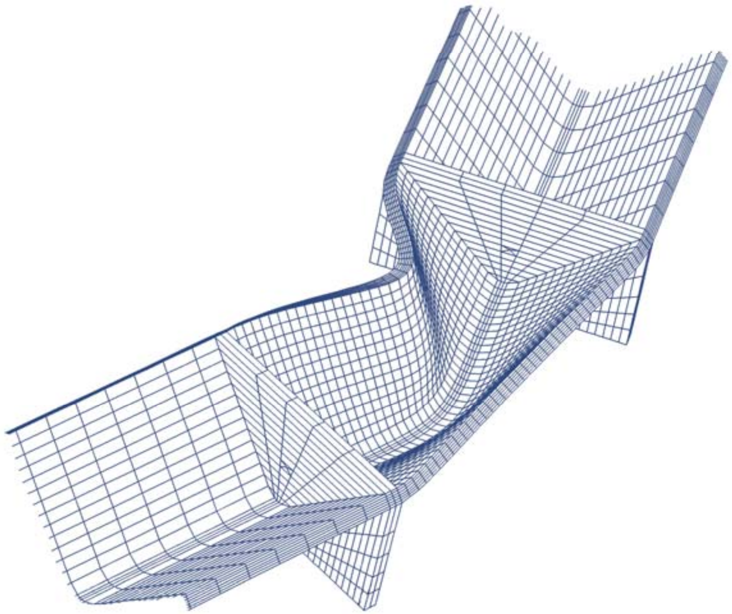
The Finite Element Method for the Analysis of Non-Linear and Dynamic Systems



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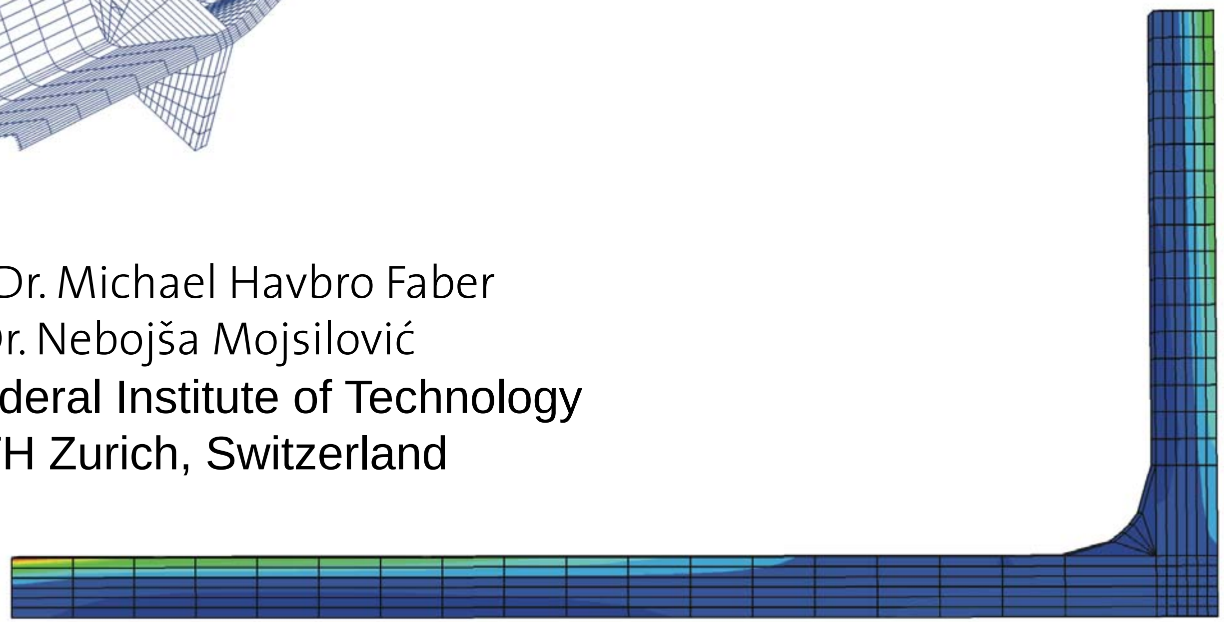
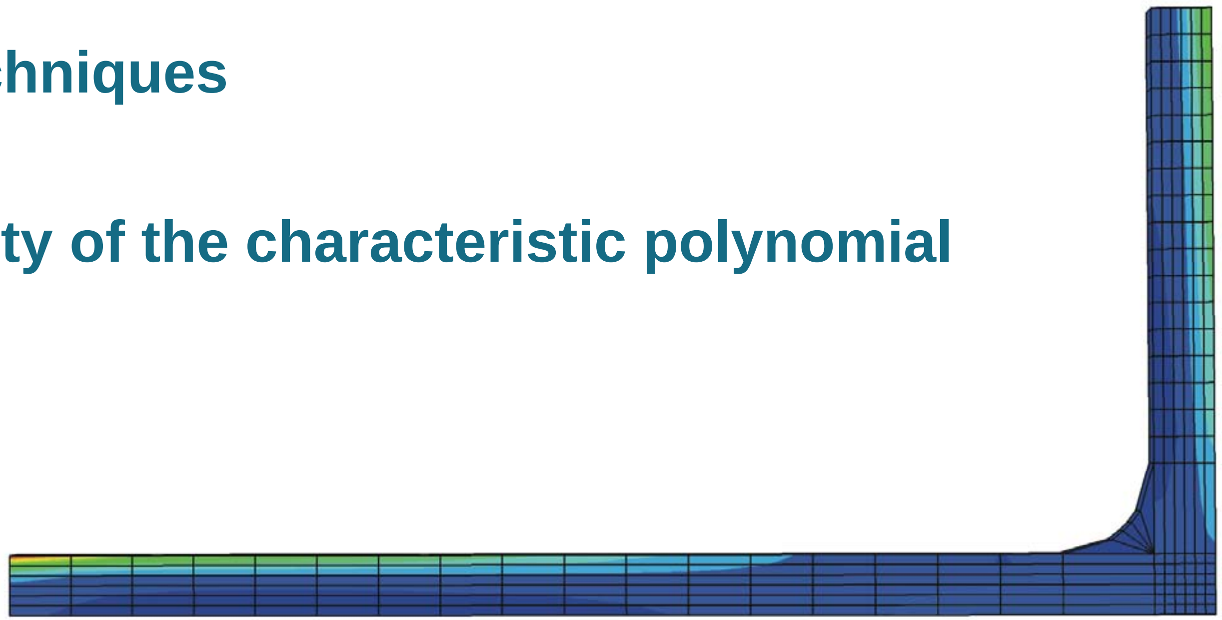


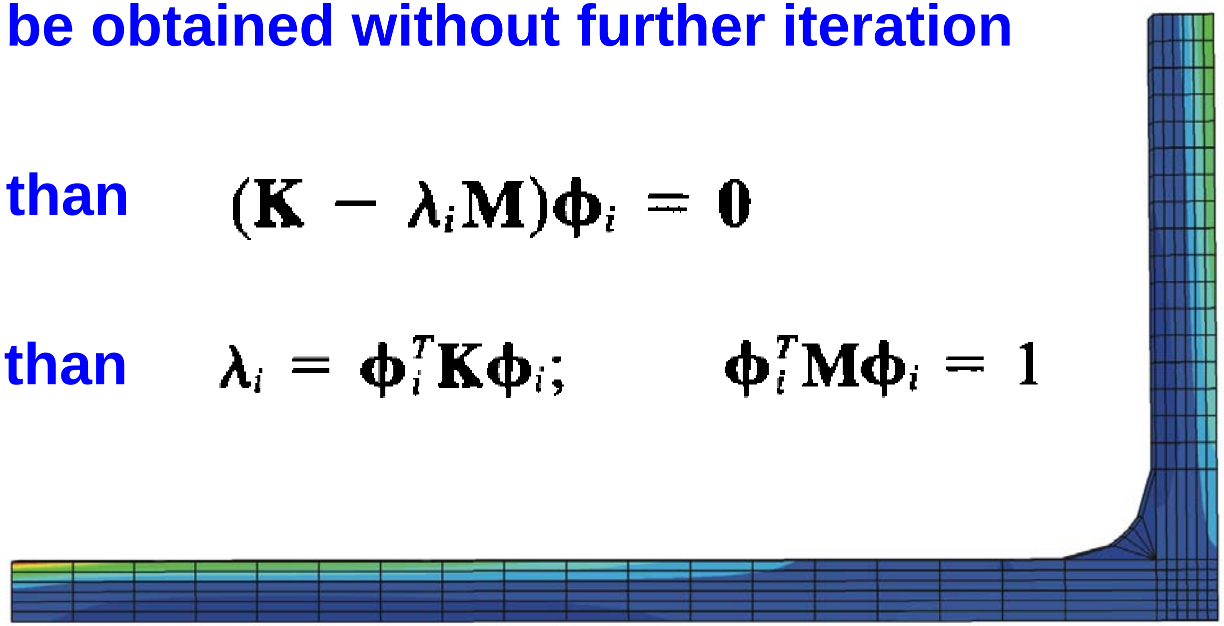
Contents of Today's Lecture

- **Solution methods for eigenproblems** $$\mathbf{K}\boldsymbol{\phi} = \lambda \mathbf{M}\boldsymbol{\phi}$$
- **Vector iteration methods**
- **Transformation methods**
- **Polynomial iteration techniques**
- **Sturm sequence property of the characteristic polynomial**



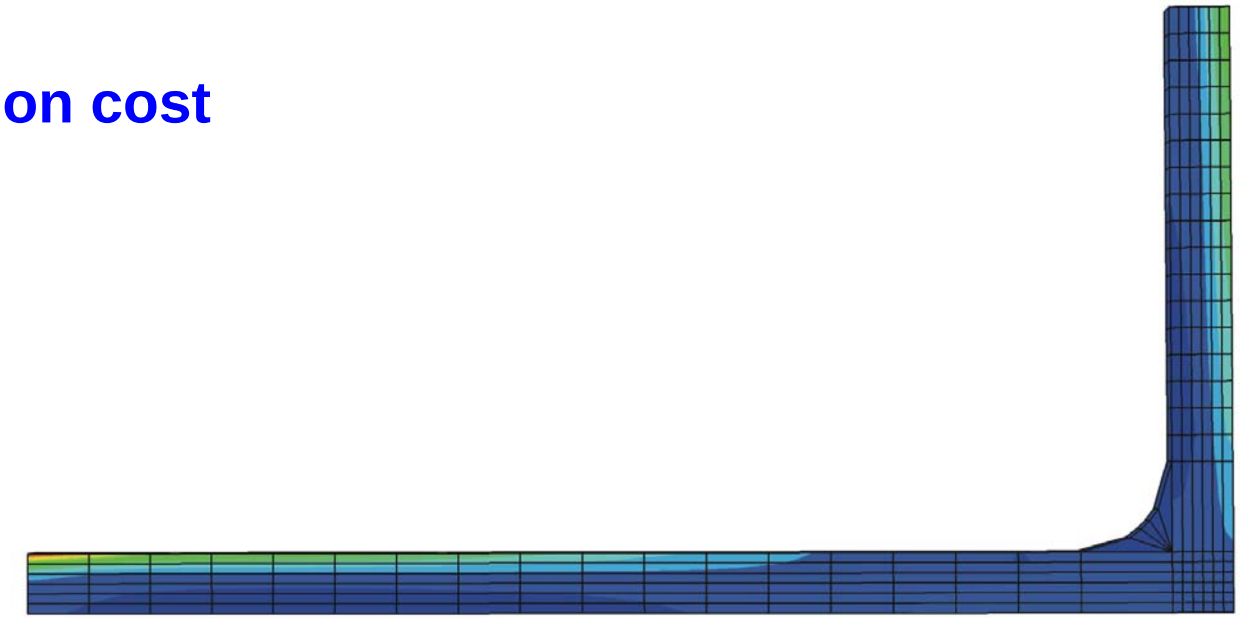
Solution methods for eigenproblems

- All solution methods are iterative since we are calculating the roots of the polynomial p(λ), which has order of K and M
- There are no implicit formulas when the order of p is higher than 4
- Iteration is needed in solution of an eigenpair (λ_i,ϕ_i); knowing one of them the other one can be obtained without further iteration
- If λ_i solved by iteration, than $$(\mathbf{K} - \lambda_i \mathbf{M})\boldsymbol{\phi}_i = \mathbf{0}$$
- If ϕ_i solved by iteration, than $$\lambda_i = \boldsymbol{\phi}_i^T \mathbf{K} \boldsymbol{\phi}_i; \qquad \boldsymbol{\phi}_i^T \mathbf{M} \boldsymbol{\phi}_i = 1$$



Effectiveness of a solution method

- Depends on two factors
- Firstly, on the possibility of a reliable use of the procedure, i.e. for well defined K and M matrices the solution is always obtained to the required precision without solution break-down
- Secondly, on the solution cost



Vector iteration methods

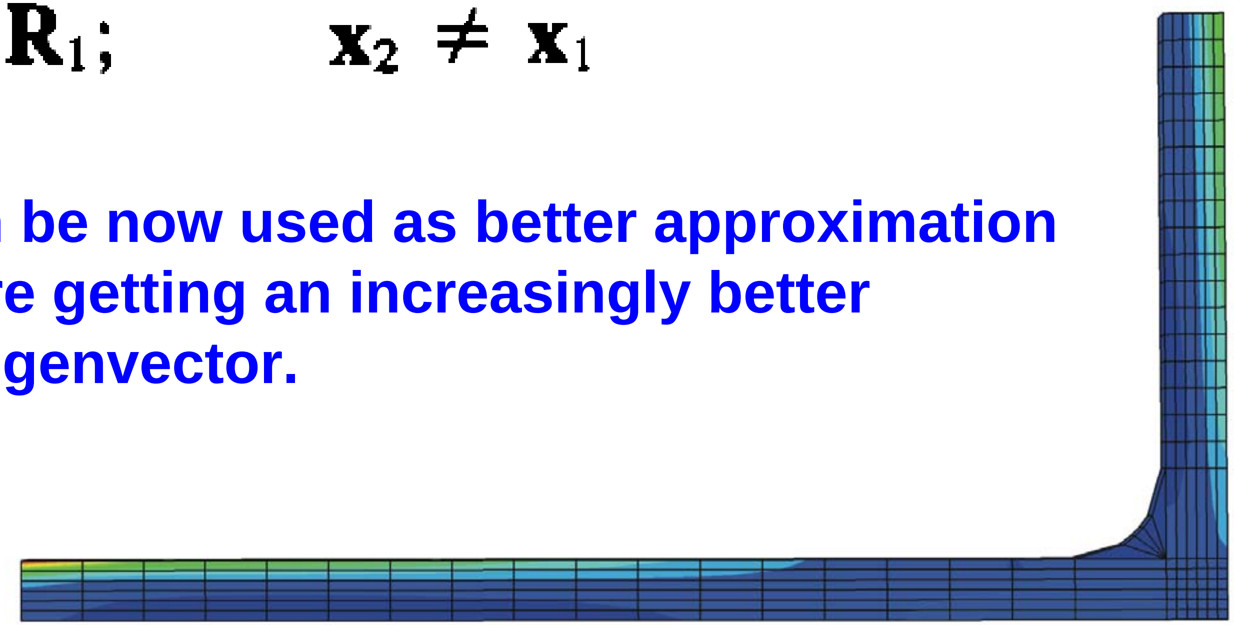
- **Problem to solve:** $$\mathbf{K}\boldsymbol{\phi} = \lambda \mathbf{M}\boldsymbol{\phi}$$

- **Assuming x_1 for ϕ and setting λ=1 we obtain** $$\mathbf{R}_1 = (1)\mathbf{M}\mathbf{x}_1$$

- **Now we can use the equilibrium equation**

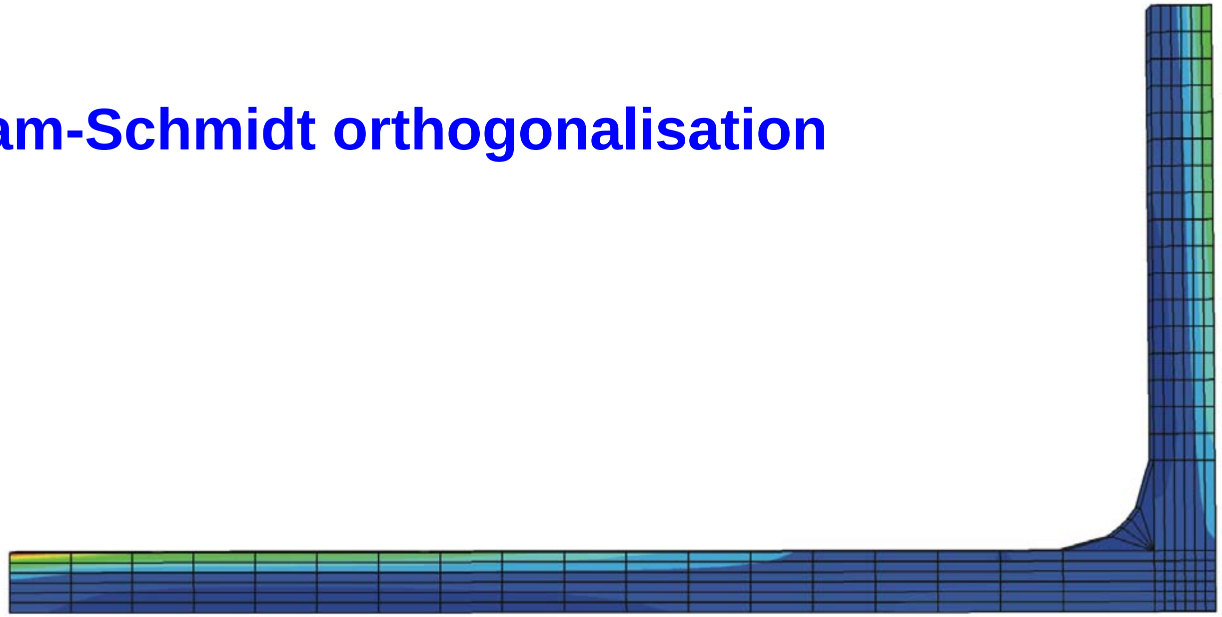
$$\mathbf{K}\mathbf{x}_2 = \mathbf{R}_1; \qquad \mathbf{x}_2 \neq \mathbf{x}_1$$

- **We obtain x_2 which can be now used as better approximation for x_1; in this way we are getting an increasingly better approximation for an eigenvector.**



Vector iteration methods

- Inverse iteration
- Forward iteration
- Rayleigh quotient iteration
- Matrix deflation and Gram-Schmidt orthogonalisation



Inverse iteration

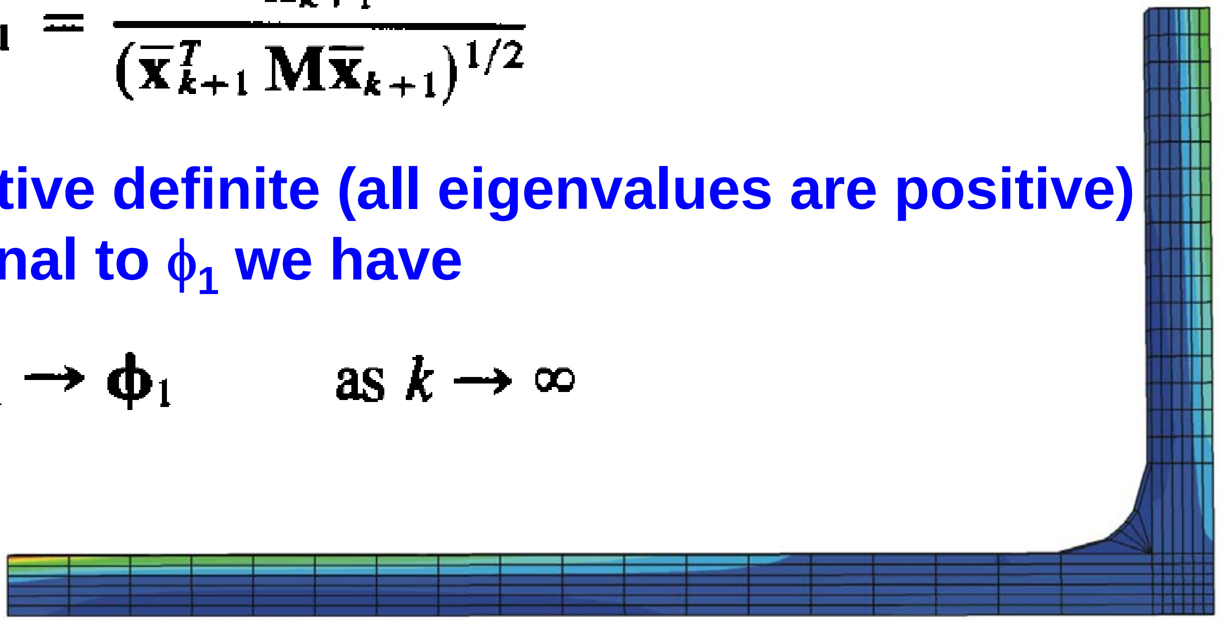
- **Used to calculate an eigenvector (and later corresponding eigenvalue)**
- **Firstly, we assume starting vector x_1 and thus in iteration step k we have**

$$\mathbf{K}\bar{\mathbf{x}}_{k+1} = \mathbf{M}\mathbf{x}_k$$

$$\mathbf{x}_{k+1} = \frac{\bar{\mathbf{x}}_{k+1}}{(\bar{\mathbf{x}}_{k+1}^T \mathbf{M}\bar{\mathbf{x}}_{k+1})^{1/2}}$$

- **Providing that K is positive definite (all eigenvalues are positive) and x_1 is not M-orthogonal to ϕ_1 we have**

$$\mathbf{x}_{k+1} \rightarrow \boldsymbol{\phi}_1 \qquad \text{as } k \rightarrow \infty$$



Forward iteration

- **Complementary to inverse iteration, as yielding the eigenvector corresponding to the largest eigenvalue**

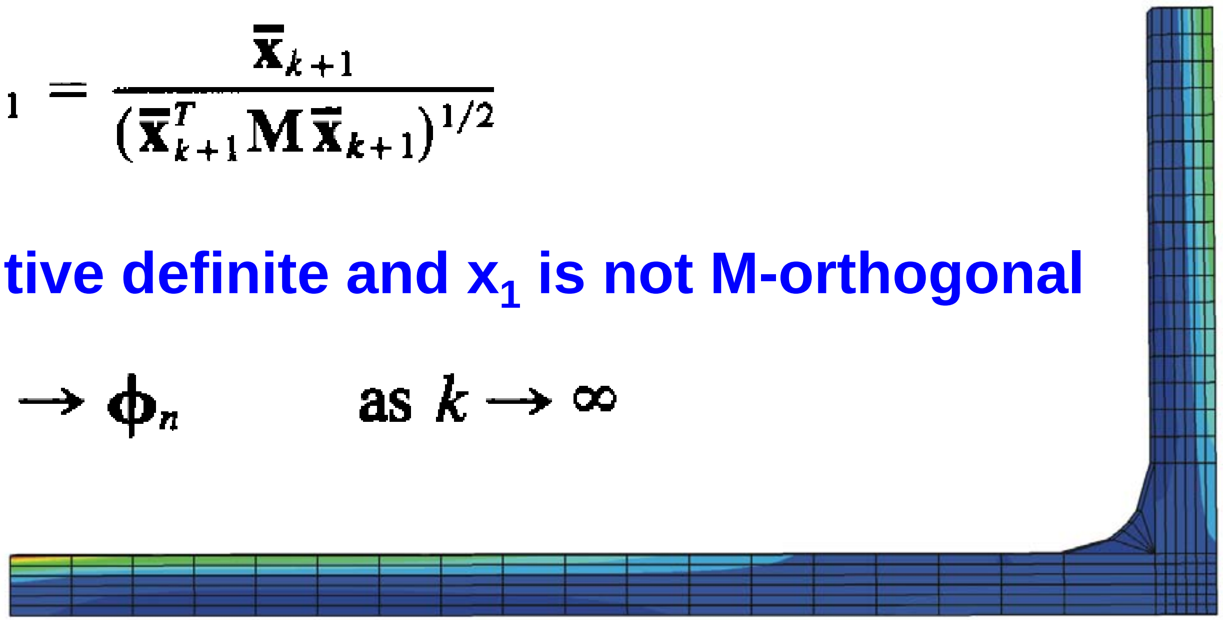
- **Firstly, we assume starting vector x_1 and thus in iteration step k we have**

$$\mathbf{M}\bar{\mathbf{x}}_{k+1} = \mathbf{K}\mathbf{x}_k$$

$$\mathbf{x}_{k+1} = \frac{\bar{\mathbf{x}}_{k+1}}{(\bar{\mathbf{x}}_{k+1}^T \mathbf{M} \bar{\mathbf{x}}_{k+1})^{1/2}}$$

- **Providing that M is positive definite and x_1 is not M-orthogonal to ϕ_1 we have**

$$\mathbf{x}_{k+1} \rightarrow \boldsymbol{\phi}_n \qquad \text{as } k \rightarrow \infty$$



Convergence of the inverse and forward iteration

- **The convergence of both procedures can be proved**
- **Convergence rate can be improved by shifting, see also Ch. 10.2.3 (we perform a shift ρ on K matrix in order to accelerate the calculations of the required eigensystem)**
- **Additionally, a shift can be used to obtain convergence in inverse iteration procedure when K is positive semi definite (all eigenvalues are greater or equal zero) and in forward iteration procedure when M is diagonal with some zero diagonal elements**