The Finite Element Method II

Dynamical finite element Calculations

solution of dynamical non-linear problems

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solution of non-linear dynamical response:

1. The incremental formulations in Charpter 6

2. The iterative solution procedures in Section 8.4

3. The time integration algorithms in Charpter 9

Aims of this lecture:

Summarize how these above procedures are employed together in a nonlinear dynamic analysis.

Explicit Integration

The central difference operator - the most common explicit time integration operator in nonlinear dynamic analys

For each discrete time step (damping matrix neglected)

$$\mathbf{M}\,{}^{t}\ddot{\mathbf{U}} = {}^{t}\mathbf{R} - {}^{t}\mathbf{F} \qquad (9.103)$$

$${}^{t}\ddot{\mathbf{U}} = \frac{1}{\Delta t^2}\left({}^{t-\Delta t}\mathbf{U} - 2\,{}^{t}\mathbf{U} + {}^{t+\Delta t}\mathbf{U}\right) \qquad (9.3)$$

Displacement at time $t + \Delta t$ solution: by central difference approximation (9.3)subtitute for ${}^{t}\ddot{\mathbf{U}}$.

Explicit Integration

central difference approximation

for $${}^{t}\ddot{\mathbf{U}} = \frac{1}{\Delta t^2}\left({}^{t-\Delta t}\mathbf{U} - 2\,{}^{t}\mathbf{U} + {}^{t+\Delta t}\mathbf{U}\right) \qquad (9.3)$$

in linear analysis, ${}^{t-\Delta t}\mathbf{U}$ and ${}^{t}\mathbf{U} \longrightarrow {}^{t+\Delta t}\mathbf{U}$

•Main advantage:

•for M is a diagonal matrix , ${}^{t+\Delta t}\mathbf{U}$ solition does not involve a triangular factorization of a coefficient matrix

•shortcoming:

•The severe time step restriction

The severe time step restriction for stability

Condition must be satisfied:

$$\Delta t \le \Delta t_{cr} = T_n / \pi$$

where T_n is the smallest period in the finite element system, Δt_{cr} is the critical time step.

Reason: Compared with a linear system

for each time step, the stiffness properties change (remain constant in a linear analysis) $\longrightarrow$ material or geometric conditions changes $\longrightarrow$ the evaluation of ${}^{t}\mathbf{F}$ $\longrightarrow$ T_n is not a constant during the response calculation $\longrightarrow$ therefore Δt need to be decreased

Explicit Integration

Example : for the simple one degree of freedom spring-mass, in a few successive solution steps, $\Delta t \geq \Delta t_{cr}$ slightly.

The solution may not show an obvious instability;
A significant error is accumulated over the solution steps;
The solution quickly "blows up".

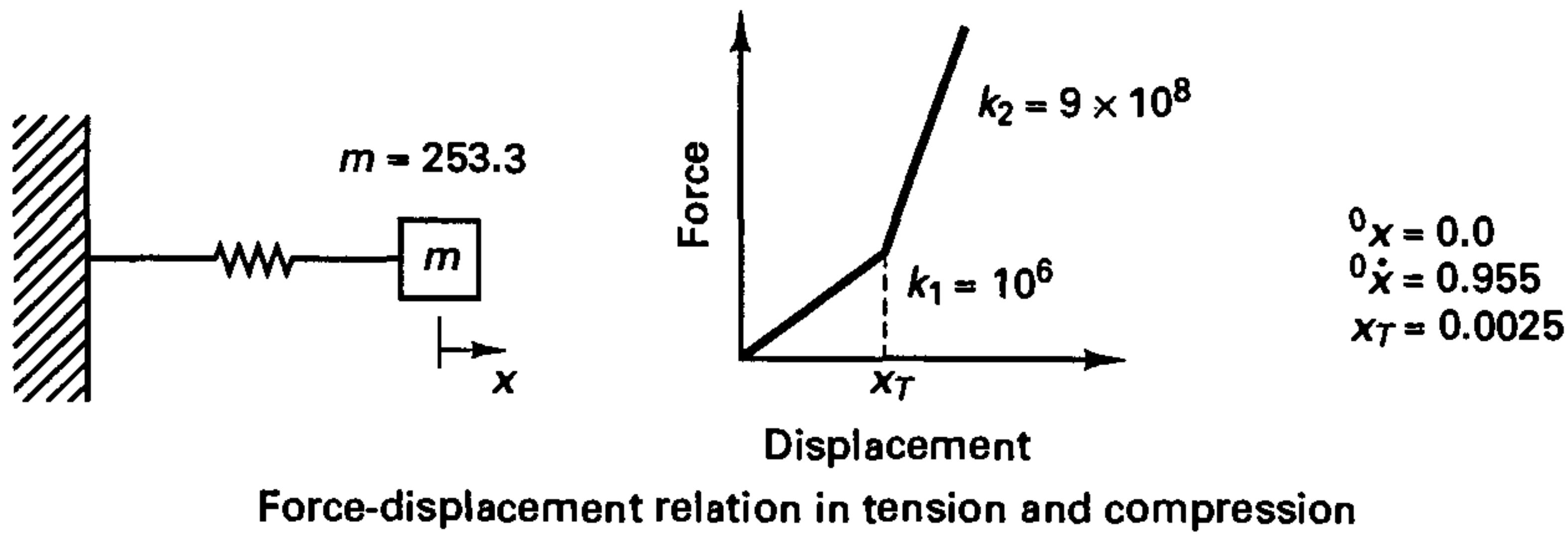
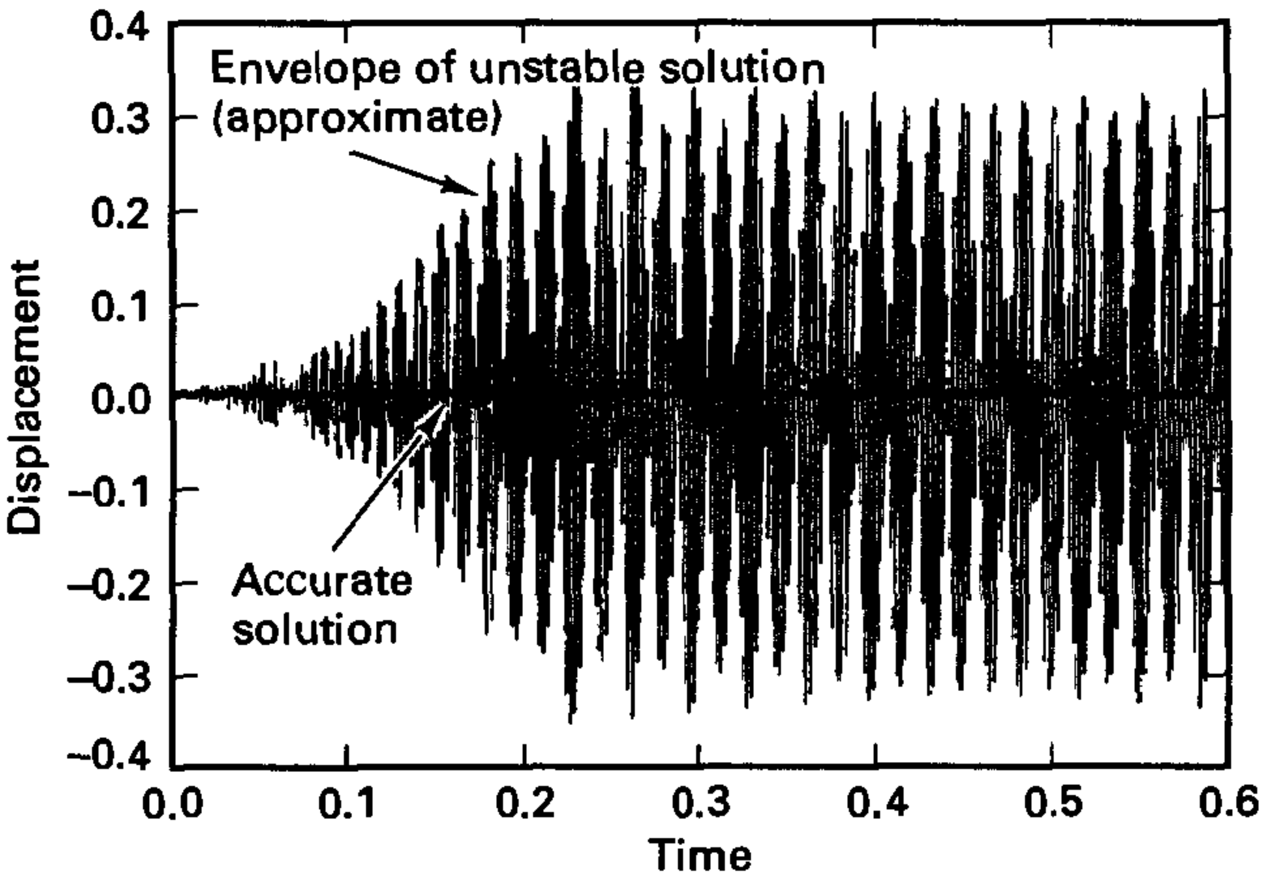


Figure 9.8 Response of bilinear elastic system as predicted using the central difference method; $\Delta t_{cr} = 0.001061027$; the accurate response with displacement $\ll 0.1$ was calculated with $\Delta t = 0.000106103$; the "unstable" response was calculated with $\Delta t = 0.00106103$.

Explicit Integration

Phenomenon

The solution may not show an obvious instability;
A significant error is accumulated over the solution steps;
The solution quickly "blows up".



Implicit Integration

A common technique :the trapezoidal rule of time integration, which is Newmark's method with $\delta = \frac{1}{2}$ and $\alpha = \frac{1}{4}$.

Assumptions of the trapezoidal rule

$$ {}^{t+\Delta t}\mathbf{U} = {}^{t}\mathbf{U} + \frac{\Delta t}{2}({}^{t}\dot{\mathbf{U}} + {}^{t+\Delta t}\dot{\mathbf{U}}) \qquad (9.106) $$

$$ {}^{t+\Delta t}\dot{\mathbf{U}} = {}^{t}\dot{\mathbf{U}} + \frac{\Delta t}{2}({}^{t}\ddot{\mathbf{U}} + {}^{t+\Delta t}\ddot{\mathbf{U}}) \qquad (9.107) $$

Implicit Integration

In nonlinear analysis, the equilibrium of the system at time $t + \Delta t$ requires an iteration.

Using the modified Newton-Raphson iteration, the governing equilibrium equations are(damping matrix neglected)

$$\mathbf{M}\ {}^{t+\Delta t}\ddot{\mathbf{U}}^{(k)} + {}^{t}\mathbf{K}\ \Delta\mathbf{U}^{(k)} = {}^{t+\Delta t}\mathbf{R} - {}^{t+\Delta t}\mathbf{F}^{(k-1)} \quad (9.104)$$

$${}^{t+\Delta t}\mathbf{U}^{(k)} = {}^{t+\Delta t}\mathbf{U}^{(k-1)} + \Delta\mathbf{U}^{(k)} \quad (9.105)$$

Combine the relations in (9.105) to (9.107) then substitute into (9.104) ,we have

$${}^{t}\hat{\mathbf{K}}\ \Delta\mathbf{U}^{(k)} = {}^{t+\Delta t}\mathbf{R} - {}^{t+\Delta t}\mathbf{F}^{(k-1)} - \mathbf{M}\left(\frac{4}{\Delta t^2}\left({}^{t+\Delta t}\mathbf{U}^{(k-1)} - {}^{t}\mathbf{U}\right) - \frac{4}{\Delta t}\,{}^{t}\dot{\mathbf{U}} - {}^{t}\ddot{\mathbf{U}}\right) \quad (9.109)$$

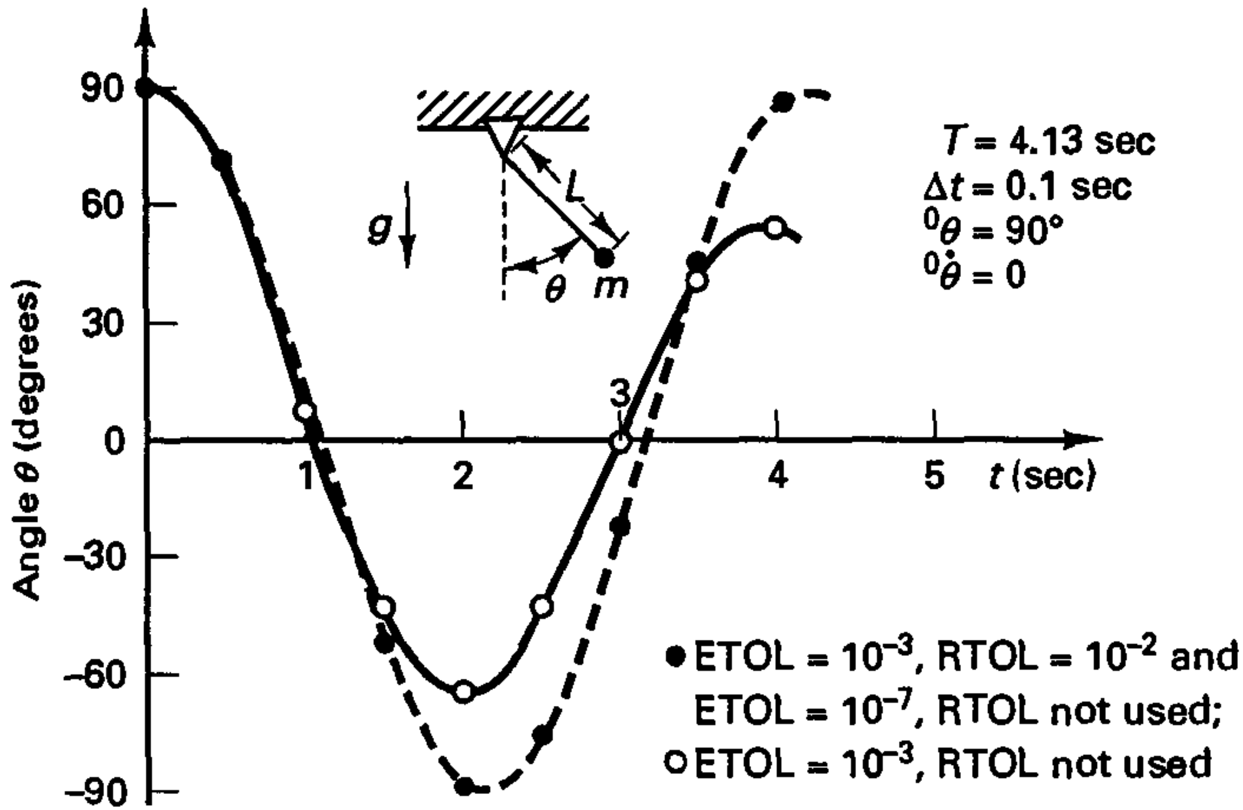
where

$${}^{t}\hat{\mathbf{K}} = {}^{t}\mathbf{K} + \frac{4}{\Delta t^2}\mathbf{M}$$

Implicit Integration

Example a pendulum idealized as a truss

The response for one period of oscillation:



Analysis of simple pendulum using trapezoidal rule

Example Phenomenon

In the first solutions of nonlinear dynamic finite element response, the equation was simple solved for k=1,and $\Delta U^{(1)}$ was accepted as an accurate approximation to the actual displacement increment.

However, any error at a particular time directly affects the solution at any subsequent time, so the iteration can be utmost important.

In summary, for a nonlinear dynamic analysis using implicit time integration

- Employ an operator –unconditionally stable in linear analysis (trapezoidal rule)

- Equilibrium iterations with tight enough convergence tolerance

- Select the time step size base on the guideline in Section 9.4.4 and on the fact convergence must be achieved

Mode superposition

The essence in considering linear analysis:

Element nodal point degrees of freedom —Transformation→ generalized degrees of freedom of the vibration mode shape

The same basic principle

The dynamic equilibrium equation in the basis of the mode shape vectors decouple(assumimg proportional damping)

Mode superposition analysis - effective if only some vibration modes are excited by loading

The same basic principle

The vibration mode shapes and frequencies at time t (calculated at a previous time) – economically calculation using the subspace iteration method(Section 11.6)

However, complete Mode superposition analysis of nonlinear dynamic response – effective only when without too frequently updating stiffness matrix. So the equilibrium at time $t + \Delta t$ are

$$\mathbf{M}\, {}^{t+\Delta t}\ddot{\mathbf{U}}^{(k)} + {}^{\tau}\mathbf{K}\, \Delta\mathbf{U}^{(k)} = {}^{t+\Delta t}\mathbf{R} - {}^{t+\Delta t}\mathbf{F}^{(k-1)} \qquad k = 1, 2, \ldots \qquad (9.113)$$

where ${}^{\tau}\mathbf{K}$ is the stiffness matrix corresponding to the configuration at some previous time τ .

Mode superposition

We use

$$ {}^{t+\Delta t}\mathbf{U} = \sum_{i=r}^{s} \boldsymbol{\phi}_i \, {}^{t+\Delta t}x_i \qquad (9.114)$$

where is the ith generalized modal displacement at time $t + \Delta t$,

and

$$ {}^{\tau}\mathbf{K}\boldsymbol{\phi}_i = \omega_i^2 \mathbf{M}\boldsymbol{\phi}_i; \qquad i = r, \ldots, s \qquad (9.115)$$

where $\omega_i, \boldsymbol{\phi}_i$ are free –vibration frequencies and mode shape vectors of the system at time τ.

The equation in (9.113) are transformed by (9.114) to

$$ {}^{t+\Delta t}\ddot{\mathbf{X}}^{(k)} + \boldsymbol{\Omega}^2 \, \Delta\mathbf{X}^{(k)} = \boldsymbol{\Phi}^T\left({}^{t+\Delta t}\mathbf{R} - {}^{t+\Delta t}\mathbf{F}^{(k-1)}\right) \qquad k = 1, 2, \ldots \qquad (9.116)$$

Mode superposition

where

$$\mathbf{\Omega}^2 = \begin{bmatrix} \omega_r^2 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \; \omega_s^2 \end{bmatrix}; \qquad \mathbf{\Phi} = [\boldsymbol{\phi}_r, \ldots, \boldsymbol{\phi}_s]; \qquad {}^{t+\Delta t}\mathbf{X} = \begin{bmatrix} {}^{t+\Delta t}x_r \\ \vdots \\ {}^{t+\Delta t}x_s \end{bmatrix} \quad (9.117)$$

The r$t+\Delta t$ns in (9.116) are the equilibrium equations at time τ in the generalized modal displacement of time . The corresponding mass matrix is an identity matrix.

This solution can use the trapezoidal rule.

In general, the use of Mode superposition in a nonlinear dynamic analysis can be effective if

•A relatively few mode shapes need to be consider in the analysis

•For example earthquake response and vibration excitation

Method Of Iteration

For a given nonlinear dynamic equation:

$$m\ddot{x} + c\dot{x} + kx \pm \mu x^3 = F\cos\omega t \qquad (1)$$

Which represents a mass on a cubic spring, excited harmonically. The $\pm$ signified a hardening or softening spring. The time t appears explicitly in the forcing term.

Given damping is zero,

$$\ddot{x} + \omega_n^2 x \pm \mu x^3 = F\cos\omega t \qquad (2)$$

to seek the steady state harmonic solution, use the method of iteration- successive approximation.

Method Of Iteration

Process(Duffing,1918)

The first assumed solution:

$$x_0 = A\cos\omega t \tag{3}$$

Substitute into the differential equation (2),

$$\ddot{x} = (-\omega_n^2 A\cos\omega t \mp \frac{3}{4}\mu A^3 + F)\cos\omega t \mp \frac{1}{4}\mu A^3 \cos 3\omega t \tag{4}$$

set the constants of integration to zero if the solution is to be harmonic with period $\tau = 2\pi/\omega$. Thus, to get the solution

$$x_1 = \frac{1}{\omega^2}(\omega_n^2 A \pm \frac{3}{4}\mu A^3 - F)\cos\omega t \mp \cdots \tag{5}$$

Repeat…

Process

If the first and the second approximations are reasonable solutions to the problem, then the coefficients of $\cos\omega t$ must not differ greatly. Therefore, by equating these confidents,

$$A = \frac{1}{\omega^2}(\omega_n^2 A \pm \frac{3}{4}\mu A^3 - F) \tag{6}$$

Which may be solved for

$$\omega^2 = \omega_n^2 \pm \frac{3}{4}\mu A^3 - \frac{F}{A} \tag{7}$$

for $\mu \neq 0$(the nonlinear parameter) ,the frequency ω is a function of μ ,F and A.

Method Of Iteration

Example solve for the period of the linear equation

$$\ddot{x}+\omega_n^2 x = F\cos\omega t \qquad \text{(a)}$$

With initial condition $x(0)=A$ and $\dot{x}(0)=0$.

Assume for the first solution $x=1$,

$$\ddot{x}=-\omega_n^2 1$$

After integrate

$$x(t)=A-\omega_n^2\frac{t^2}{2} \qquad \text{(b)}$$

letting $t=t_1$ at a quarter cycle and noting $x(t_1)=0$,we can get

$$x=A\left(1-\frac{t^2}{t_1^{\,2}}\right) \qquad \text{(c)}$$

Now substitute (c) into (a) and repeat the process

Method Of Iteration

Example solve for the period of the linear equation

$$\dot{x}(t) = -\omega_n^2 A \int_0^{\xi} (1 - \frac{\xi^2}{t_1^{\ 2}}) d\xi$$

$$= -\omega_n^2 A (1 - \frac{\xi^2}{3t_1^{\ 2}})$$

(d)

$$x(t) = A - \omega_n^2 A \int_0^{t} (\xi - \frac{\xi^3}{3t_1^{\ 2}}) d\xi$$

$$= A - \omega_n^2 A (\frac{t^2}{2} - \frac{t^4}{12t_1^{\ 2}})$$

Method Of Iteration

Example solve for the period of the linear equation

Next let $t = t_1$ and $x(t_1) = 0$.

$$0 = A - \omega_n^2 (\frac{t_1^{\ 2}}{2} - \frac{t_1^{\ 4}}{12})$$

Solving for t_1 we obtain

$$t_1 = \frac{1}{\omega}\sqrt{\frac{12}{5}} = \frac{\tau}{2\pi}\sqrt{\frac{12}{5}} = \frac{\tau}{4.05}$$

After two iterations ,the value of t_1 is found near $\tau/4$.

Other methods for solving nonlinear equations

Isoclines method $$\frac{dy}{dx} = -\frac{f(x,y)}{y} = \phi(x,y)$$

Delta method $$\ddot{x} + f(\dot{x}, x, t) - \omega_0^2 x + \omega_0^2 x = 0$$

Perturbation method $$\ddot{x} + \omega_n^2 x + \mu x^3 = 0$$

To seek a solution in the form of an infinite series of the perturbation parameter as

$$x = x_0(t) + \mu x_1(t) + \mu x_2(t) + \cdots$$

Self-excited oscillations $$m\ddot{x} + c\dot{x} + kx = F(\dot{x})$$

Thank you!