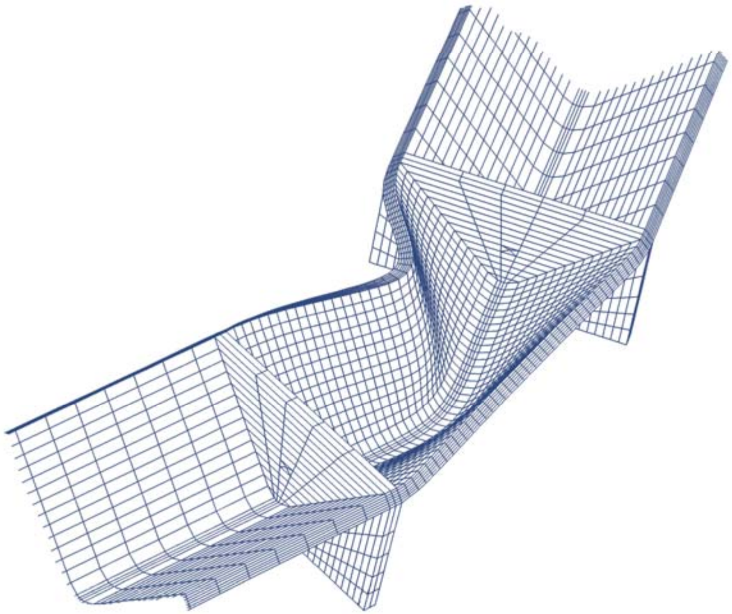
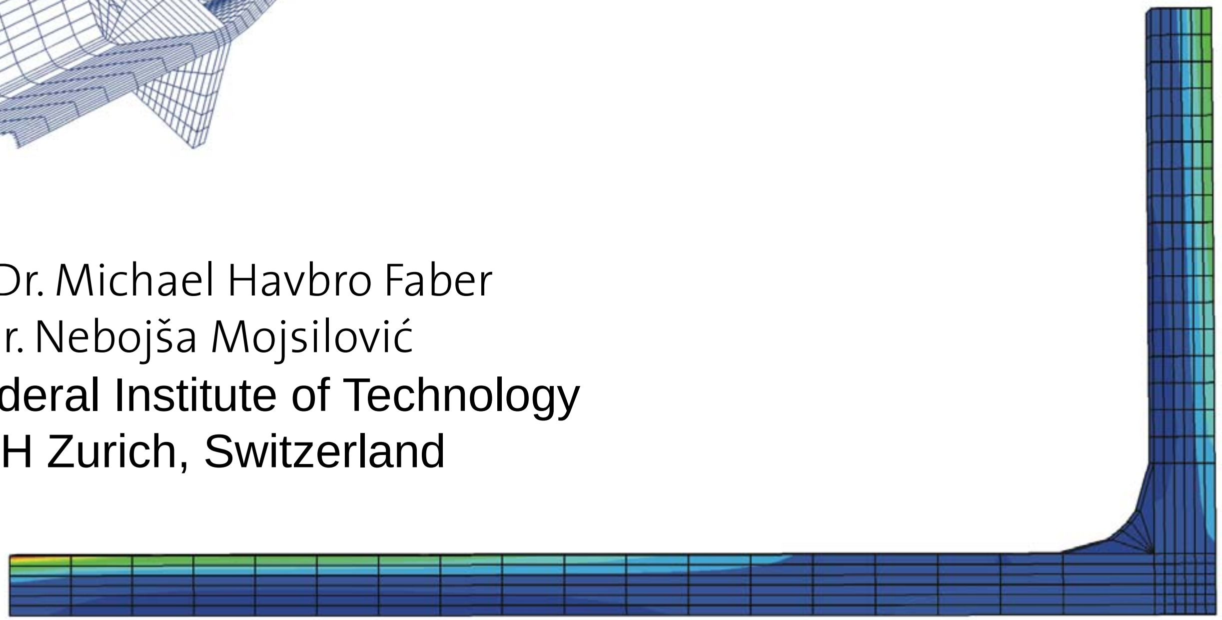
The Finite Element Method for the Analysis of Non-Linear and Dynamic Systems

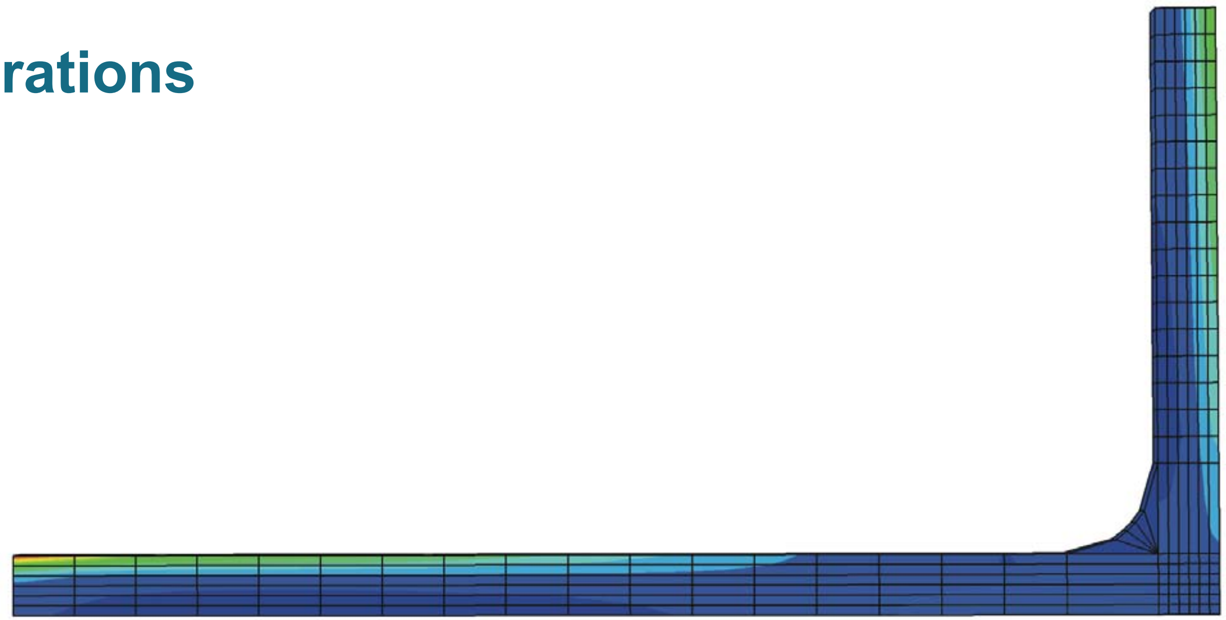


Prof. Dr. Michael Havbro Faber
Dr. Nebojša Mojsilović
Swiss Federal Institute of Technology
ETH Zurich, Switzerland



Contents of Today's Lecture

- **Analysis of direct integration methods**
- **Stability analysis (liner analysis)**
- **Accuracy analysis (liner analysis)**
- **Some practical considerations**
- **Examples**

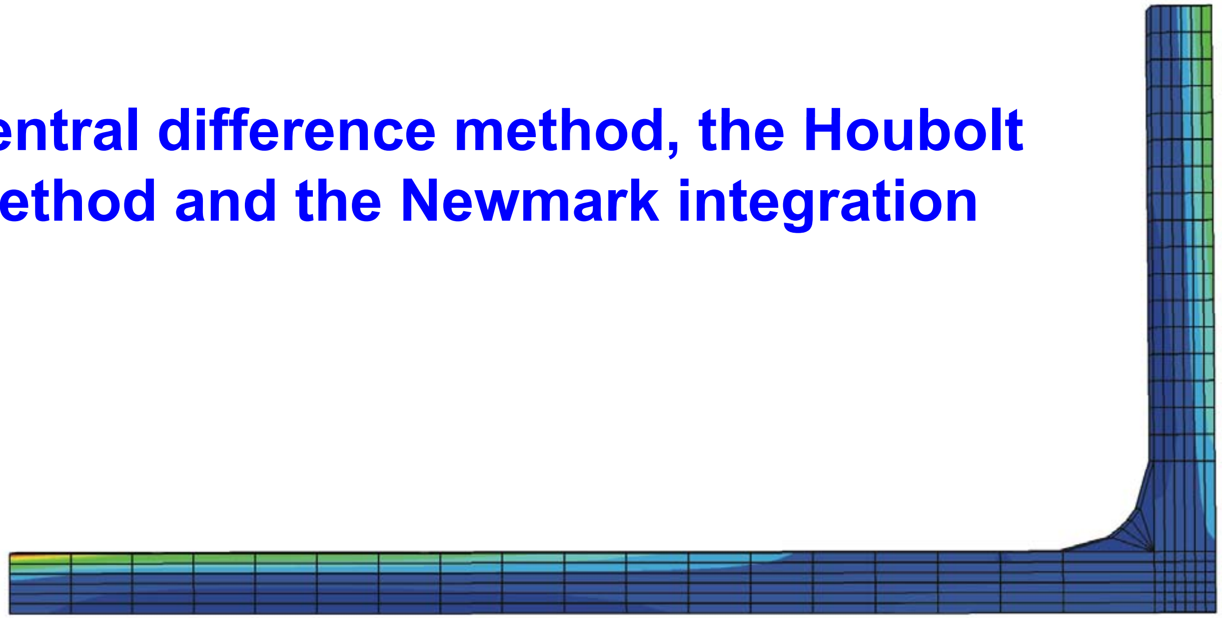


Analysis of direct integration methods

- **Dynamic equilibrium equation**

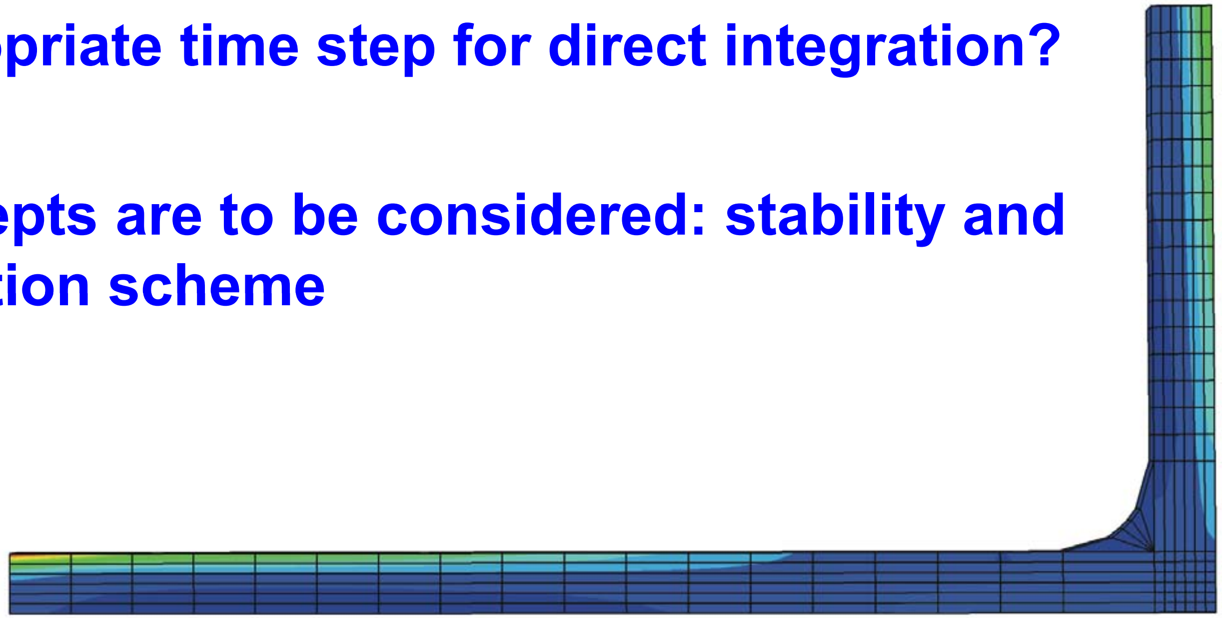
$$\mathbf{M}\ddot{\mathbf{U}}(t) + \mathbf{C}\dot{\mathbf{U}}(t) + \mathbf{K}\mathbf{U}(t) = \mathbf{R}(t)$$

- **Solution procedures: mode superposition and direct integration**
- **Integration schemes: central difference method, the Houbolt method, the Wilson θ method and the Newmark integration procedure**



Analysis of direct integration methods

- Key to all integration procedures: time step Δt
- Small enough to obtain the accurate solution
- Large enough to save computational time
- How to select the appropriate time step for direct integration?
- Two fundamental concepts are to be considered: stability and accuracy of the integration scheme



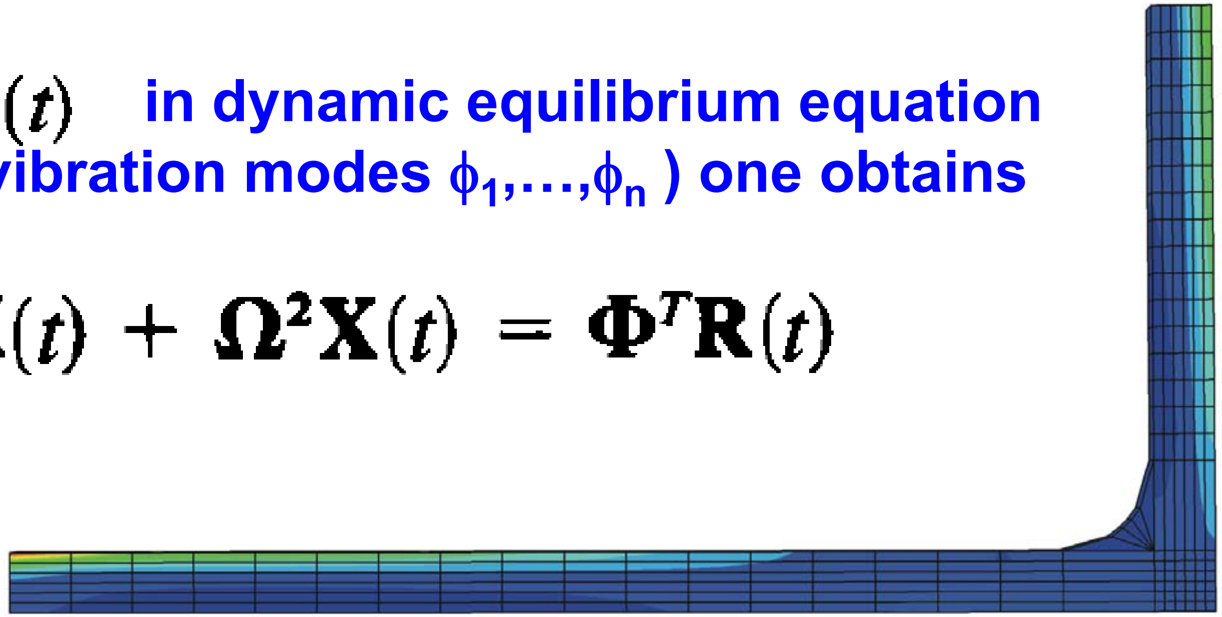
Analysis of direct integration methods

- Relation between mode superposition and direct integration is of interest
- Mode superposition: change of basis from the nodal displacements to the basis of eigenvectors of generalised eigenproblem

$$\mathbf{K}\boldsymbol{\phi} = \omega^2 \mathbf{M}\boldsymbol{\phi}$$

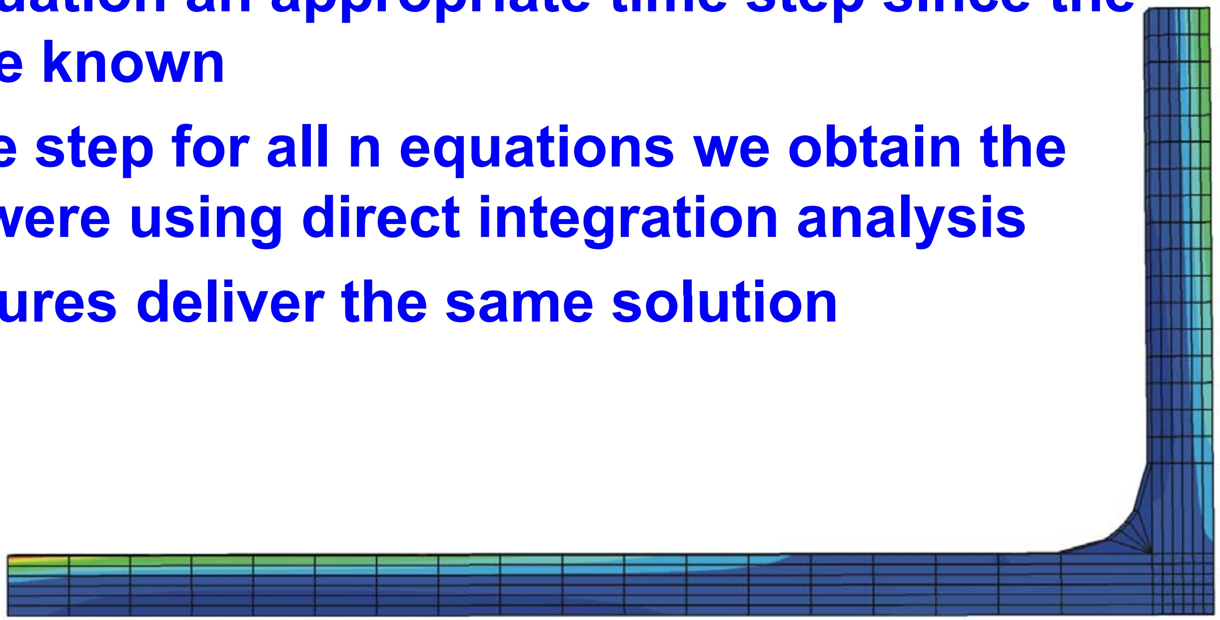
- Setting $\mathbf{U}(t) = \boldsymbol{\Phi}\mathbf{X}(t)$ in dynamic equilibrium equation (columns in Φ are free-vibration modes $\phi_1,\ldots,\phi_n$) one obtains

$$\ddot{\mathbf{X}}(t) + \boldsymbol{\Delta}\dot{\mathbf{X}}(t) + \boldsymbol{\Omega}^2\mathbf{X}(t) = \boldsymbol{\Phi}^T\mathbf{R}(t)$$



Analysis of direct integration methods

- where Ω^2 is diagonal matrix with free-vibration frequencies squared $\omega^2_1,\ldots,\omega^2_n$, Δ is also a diagonal matrix of damping $\Delta = \text{diag}(2\omega_i\xi_i)$, ξ_i being the damping ratio in i_{th} mode
- The obtained equation system consists of n uncoupled equations and can be solved using one of the direct integration procedures we already mentioned
- We can use for each equation an appropriate time step since the periods of vibrations are known
- Choosing the same time step for all n equations we obtain the same solution as if we were using direct integration analysis
- In this way both procedures deliver the same solution

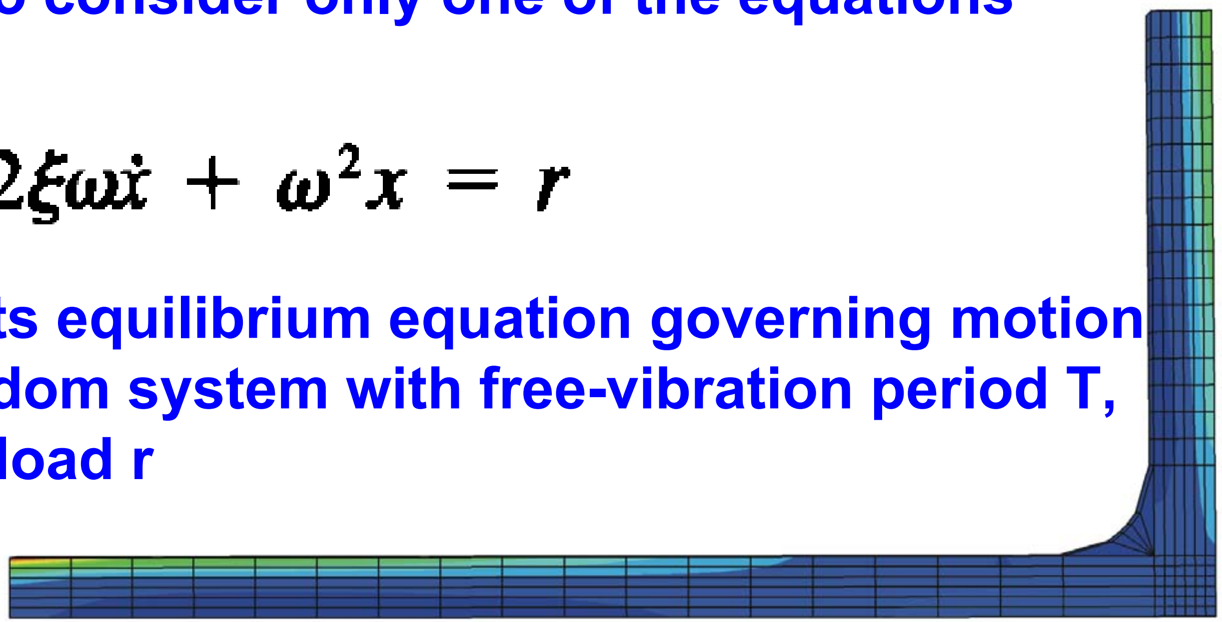


Analysis of direct integration methods

- So, to study the accuracy of direct integration we can consider the equations of mode superposition using the same time step by integration
- Thus, the variables that have to be considered in the stability and accuracy analysis of of the direct integration method are only Δt, ω_i and ξ_i for i=1,…n.
- Furthermore, we need to consider only one of the equations (since they are similar):

$$\ddot{x} + 2\xi\omega\dot{x} + \omega^2 x = r$$

- This equation represents equilibrium equation governing motion of single degree of freedom system with free-vibration period T, dumping ξ and applied load r

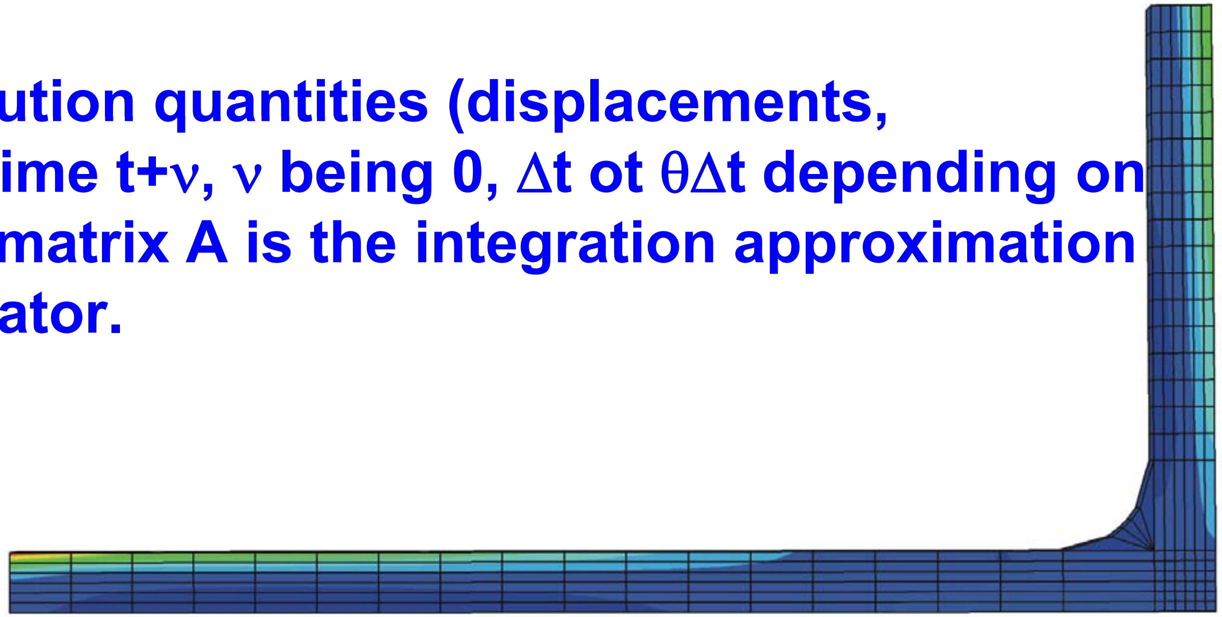


Direct integration approximation and load operators

- **Having solutions for the discrete times 0, Δt, 2Δt,…t-Δt, t the solution for the time t+Δt can be obtained using recursive relationship in the specific integration method considered**

$$ {}^{t+\Delta t}\hat{\mathbf{X}} = \mathbf{A}\,{}^{t}\hat{\mathbf{X}} + \mathbf{L}({}^{t+\nu}r) $$

- **Vectors X are storing solution quantities (displacements, velocities), r is a load at time t+ν, ν being 0, Δt ot θΔt depending on the chosen method. The matrix A is the integration approximation and vector L is load operator.**



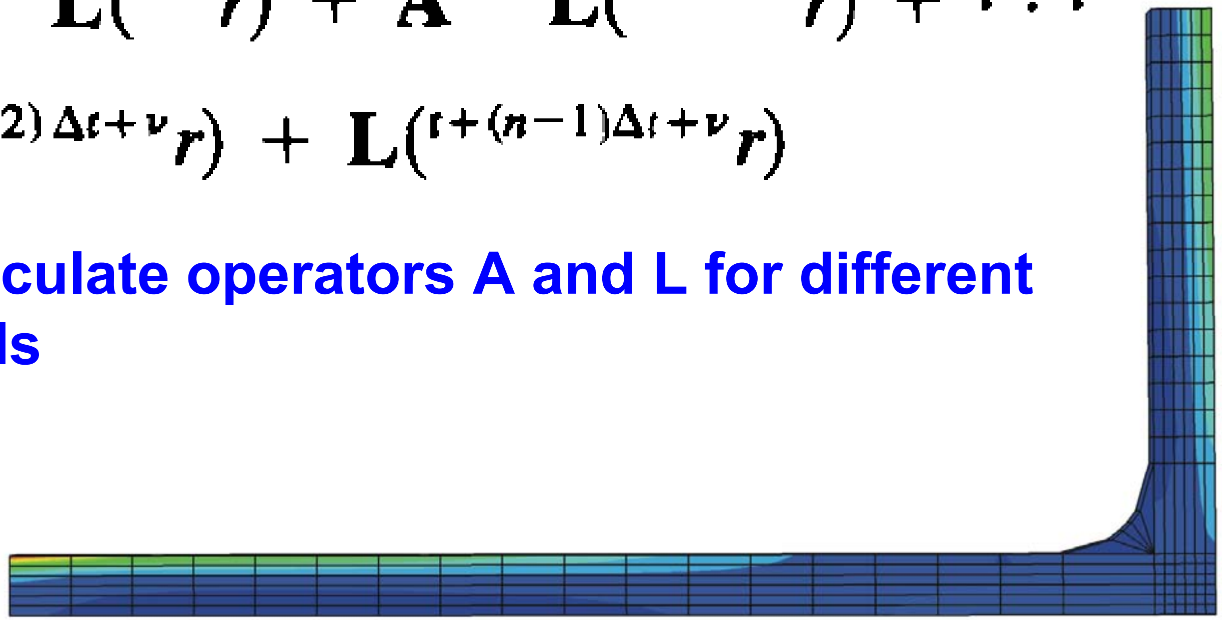
Direct integration approximation and load operators

- To study the accuracy and stability of the integration methods we will need the relation which gives the solution for the time t+nΔt and which is obtained in applying the above relation recursively:

$$ {}^{t+n\Delta t}\hat{\mathbf{X}} = \mathbf{A}^n\,{}^{t}\hat{\mathbf{X}} + \mathbf{A}^{n-1}\mathbf{L}({}^{t+\nu}\mathbf{r}) + \mathbf{A}^{n-2}\mathbf{L}({}^{t+\Delta t+\nu}\mathbf{r}) + \cdots $$

$$ + \mathbf{A}\,\mathbf{L}({}^{t+(n-2)\Delta t+\nu}\mathbf{r}) + \mathbf{L}({}^{t+(n-1)\Delta t+\nu}\mathbf{r}) $$

- Now, we proceed and calculate operators A and L for different direct integration methods



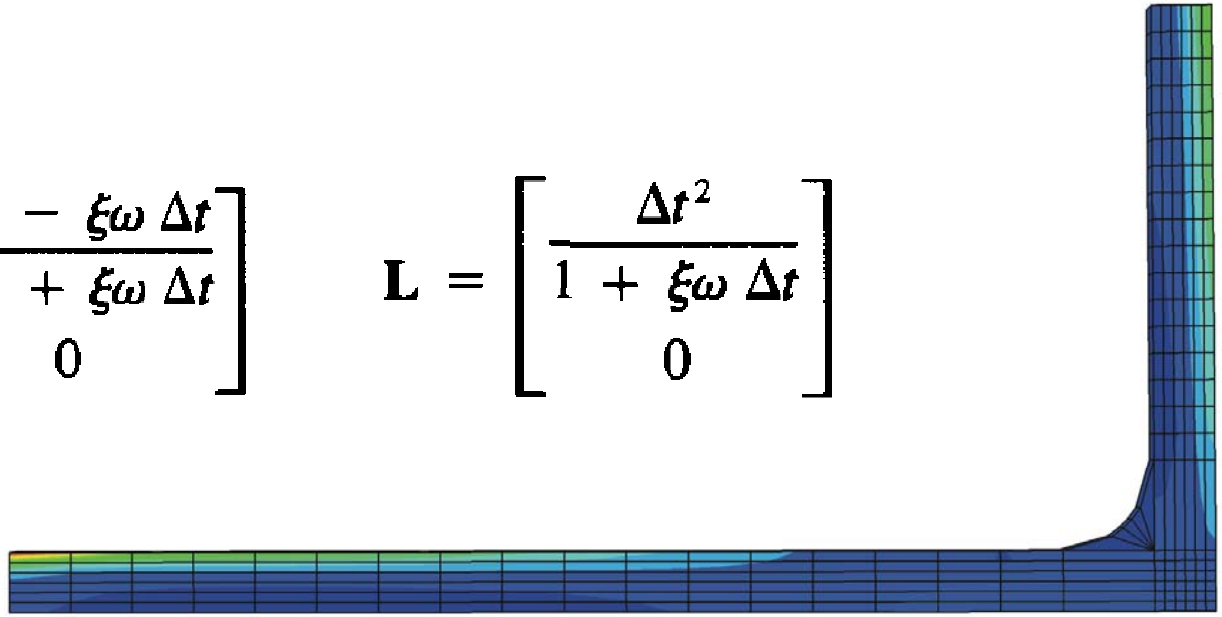
The central difference method

- We approximate acceleration and velocity at time t. The equilibrium equation is also considered in time t. Thus we obtain

$$\begin{bmatrix} {}^{t+\Delta t}x \\ {}^{t}x \end{bmatrix} = \mathbf{A} \begin{bmatrix} {}^{t}x \\ {}^{t-\Delta t}x \end{bmatrix} + \mathbf{L}\,{}^{t}r$$

- where

$$\mathbf{A} = \begin{bmatrix} \dfrac{2 - \omega^2 \Delta t^2}{1 + \xi\omega \Delta t} & -\dfrac{1 - \xi\omega \Delta t}{1 + \xi\omega \Delta t} \\ 1 & 0 \end{bmatrix} \qquad \mathbf{L} = \begin{bmatrix} \dfrac{\Delta t^2}{1 + \xi\omega \Delta t} \\ 0 \end{bmatrix}$$



The Houbolt method

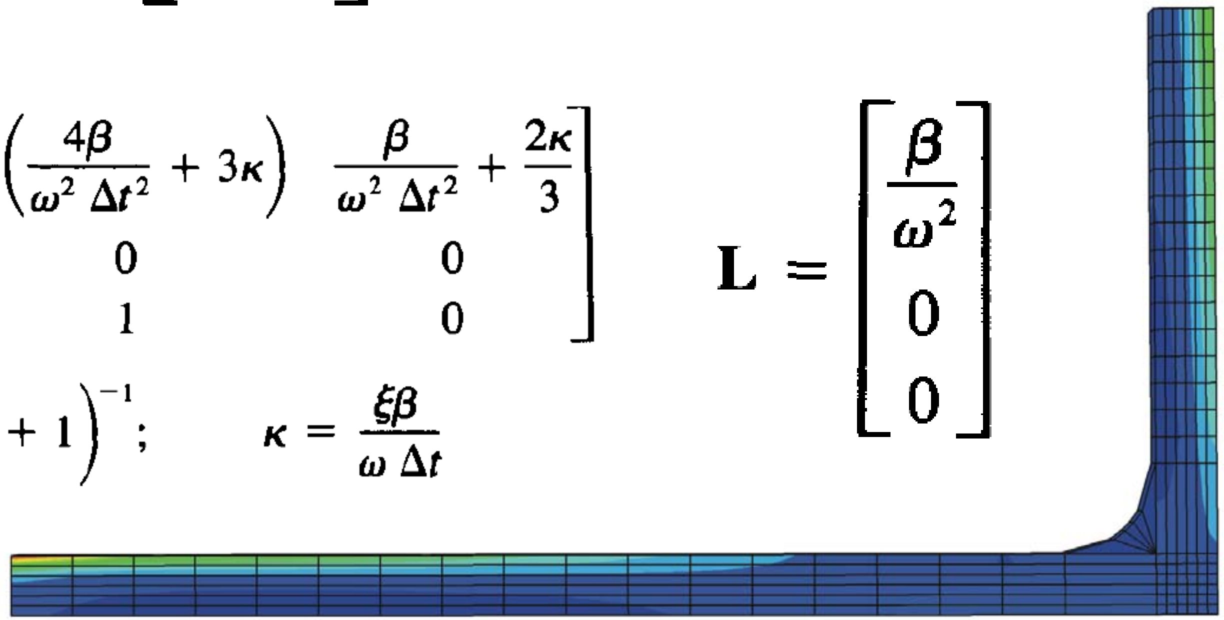
- **The equilibrium equation is considered in time t+Δt and two backward formulas are used for the acceleration and velocity at time t+Δt. Thus we obtain**

$$\begin{bmatrix} {}^{t+\Delta t}x \\ {}^{t}x \\ {}^{t-\Delta t}x \end{bmatrix} = \mathbf{A} \begin{bmatrix} {}^{t}x \\ {}^{t-\Delta t}x \\ {}^{t-2\Delta t}x \end{bmatrix} + \mathbf{L}\, {}^{t+\Delta t}r$$

- **where**

$$\mathbf{A} = \begin{bmatrix} \dfrac{5\beta}{\omega^2\,\Delta t^2} + 6\kappa & -\left(\dfrac{4\beta}{\omega^2\,\Delta t^2} + 3\kappa\right) & \dfrac{\beta}{\omega^2\,\Delta t^2} + \dfrac{2\kappa}{3} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad \mathbf{L} = \begin{bmatrix} \dfrac{\beta}{\omega^2} \\ 0 \\ 0 \end{bmatrix}$$

$$\beta = \left(\frac{2}{\omega^2\,\Delta t^2} + \frac{11\xi}{3\omega\,\Delta t} + 1\right)^{-1}; \qquad \kappa = \frac{\xi\beta}{\omega\,\Delta t}$$



The Wilson θ method

- **The equilibrium equation is considered in time t+θΔt. Thus we obtain**

$$\begin{bmatrix} {}^{t+\Delta t}\ddot{x} \\ {}^{t+\Delta t}\dot{x} \\ {}^{t+\Delta t}x \end{bmatrix} = \mathbf{A} \begin{bmatrix} {}^{t}\ddot{x} \\ {}^{t}\dot{x} \\ {}^{t}x \end{bmatrix} + \mathbf{L}\ {}^{t+\theta\Delta t}r$$

- **where**

$$\mathbf{A} = \begin{bmatrix} 1 - \dfrac{\beta\theta^2}{3} - \dfrac{1}{\theta} - \kappa\theta & \dfrac{1}{\Delta t}(-\beta\theta - 2\kappa) & \dfrac{1}{\Delta t^2}(-\beta) \\ \Delta t\left(1 - \dfrac{1}{2\theta} - \dfrac{\beta\theta^2}{6} - \dfrac{\kappa\theta}{2}\right) & 1 - \dfrac{\beta\theta}{2} - \kappa & \dfrac{1}{\Delta t}\left(-\dfrac{\beta}{2}\right) \\ \Delta t^2\left(\dfrac{1}{2} - \dfrac{1}{6\theta} - \dfrac{\beta\theta^2}{18} - \dfrac{\kappa\theta}{6}\right) & \Delta t\left(1 - \dfrac{\beta\theta}{6} - \dfrac{\kappa}{3}\right) & 1 - \dfrac{\beta}{6} \end{bmatrix} \qquad \mathbf{L} = \begin{bmatrix} \dfrac{\beta}{\omega^2\,\Delta t^2} \\ \dfrac{\beta}{2\omega^2\,\Delta t} \\ \dfrac{\beta}{6\omega^2} \end{bmatrix}$$

$$\beta = \left(\frac{\theta}{\omega^2\,\Delta t^2} + \frac{\xi\,\theta^2}{\omega\,\Delta t} + \frac{\theta^3}{6}\right)^{-1}; \qquad \kappa = \frac{\xi\beta}{\omega\,\Delta t}$$

The Newmark method

- **The equilibrium equation is considered in time t+Δt. For the velocity and displacements at time t+Δt two parameters, δ and α are to be chosen. Thus we obtain**

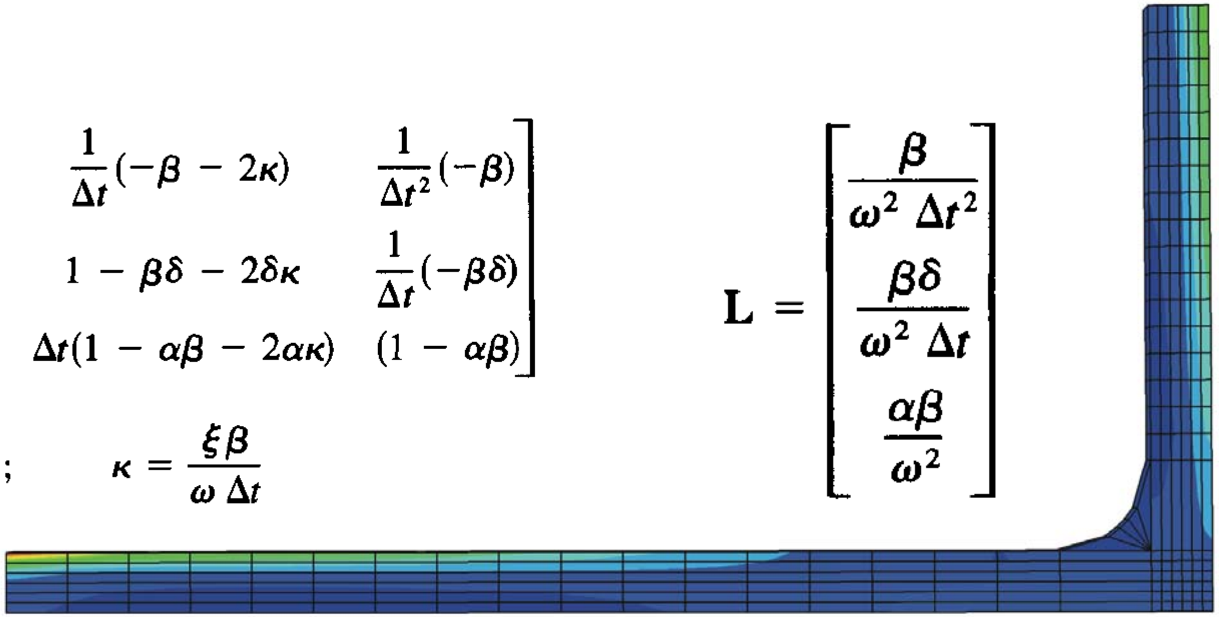
$$\begin{bmatrix} {}^{t+\Delta t}\ddot{x} \\ {}^{t+\Delta t}\dot{x} \\ {}^{t+\Delta t}x \end{bmatrix} = \mathbf{A} \begin{bmatrix} {}^{t}\ddot{x} \\ {}^{t}\dot{x} \\ {}^{t}x \end{bmatrix} + \mathbf{L}\, {}^{t+\Delta t}r$$

- **where**

$$\mathbf{A} = \begin{bmatrix} -(\frac{1}{2} - \alpha)\beta - 2(1-\delta)\kappa & \frac{1}{\Delta t}(-\beta - 2\kappa) & \frac{1}{\Delta t^2}(-\beta) \\ \Delta t[1 - \delta - (\frac{1}{2} - \alpha)\delta\beta - 2(1-\delta)\delta\kappa] & 1 - \beta\delta - 2\delta\kappa & \frac{1}{\Delta t}(-\beta\delta) \\ \Delta t^2[\frac{1}{2} - \alpha - (\frac{1}{2} - \alpha)\alpha\beta - 2(1-\delta)\alpha\kappa] & \Delta t(1 - \alpha\beta - 2\alpha\kappa) & (1 - \alpha\beta) \end{bmatrix}$$

$$\mathbf{L} = \begin{bmatrix} \dfrac{\beta}{\omega^2\,\Delta t^2} \\ \dfrac{\beta\delta}{\omega^2\,\Delta t} \\ \dfrac{\alpha\beta}{\omega^2} \end{bmatrix}$$

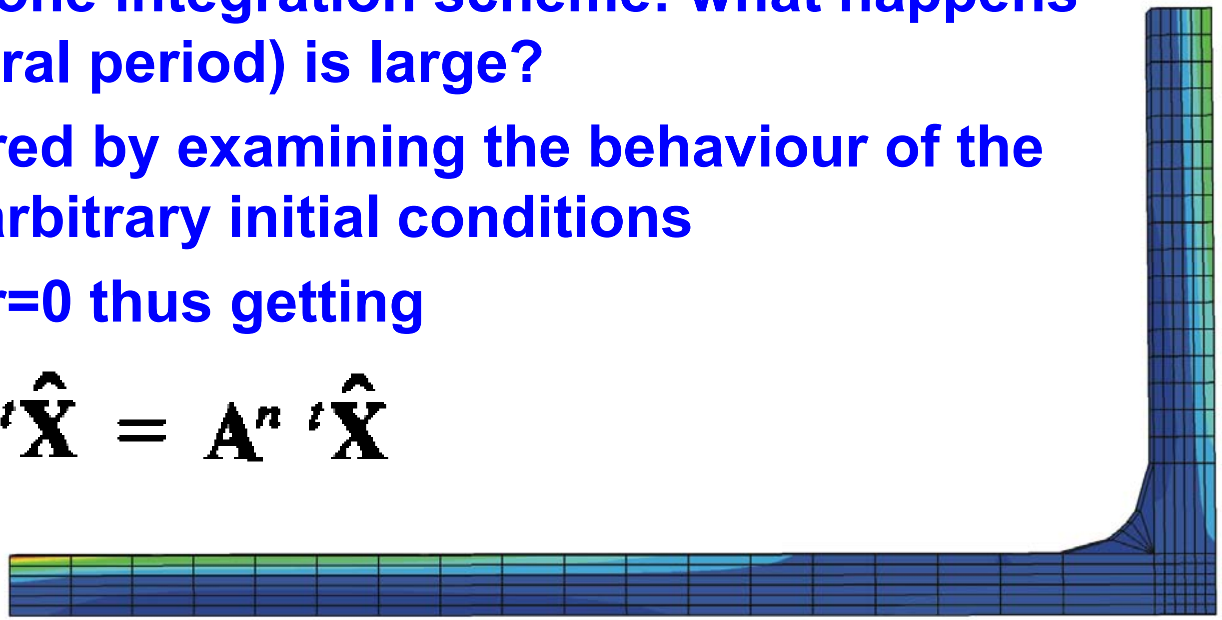
$$\beta = \left(\frac{1}{\omega^2\,\Delta t^2} + \frac{2\xi\delta}{\omega\,\Delta t} + \alpha\right)^{-1}; \qquad \kappa = \frac{\xi\beta}{\omega\,\Delta t}$$



Stability analysis

- In general, Δt should be chosen according to the smallest period in system T_n; usually $T_n/10$ or smaller
- Practically, we need to consider only first p of total n equations and accordingly we need to perform finite element idealization such that the lowest p frequencies and mode shapes are predicted accurately
- Question of stability of one integration scheme: what happens when $\Delta t/T$ (T being natural period) is large?
- This question is answered by examining the behaviour of the numerical solution for arbitrary initial conditions
- To perform this we set r=0 thus getting

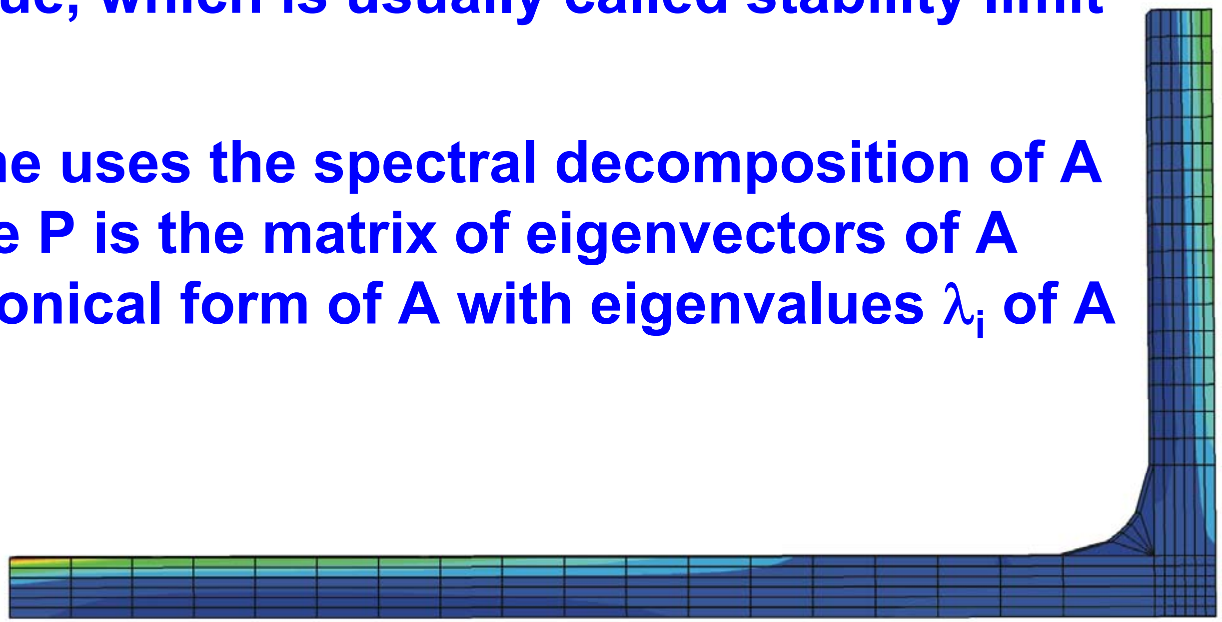
$$ {}^{t+n\Delta t}\hat{\mathbf{X}} = \mathbf{A}^n \, {}^{t}\hat{\mathbf{X}} $$



Stability analysis

- **Considering the stability of integration method we have procedures that are:**
- **Unconditionally stable – if the solution for any initial conditions does not grow without bound for any time step Δt, in particular when $\Delta t/T$ is large**
- **Conditionally stable – if the above holds only when $\Delta t/T$ is smaller or equal to a certain value, which is usually called stability limit**

- **For stability analysis one uses the spectral decomposition of A given by $A=PJP^{-1}$, where P is the matrix of eigenvectors of A and J is the Jordan canonical form of A with eigenvalues λ_i of A on its diagonal**



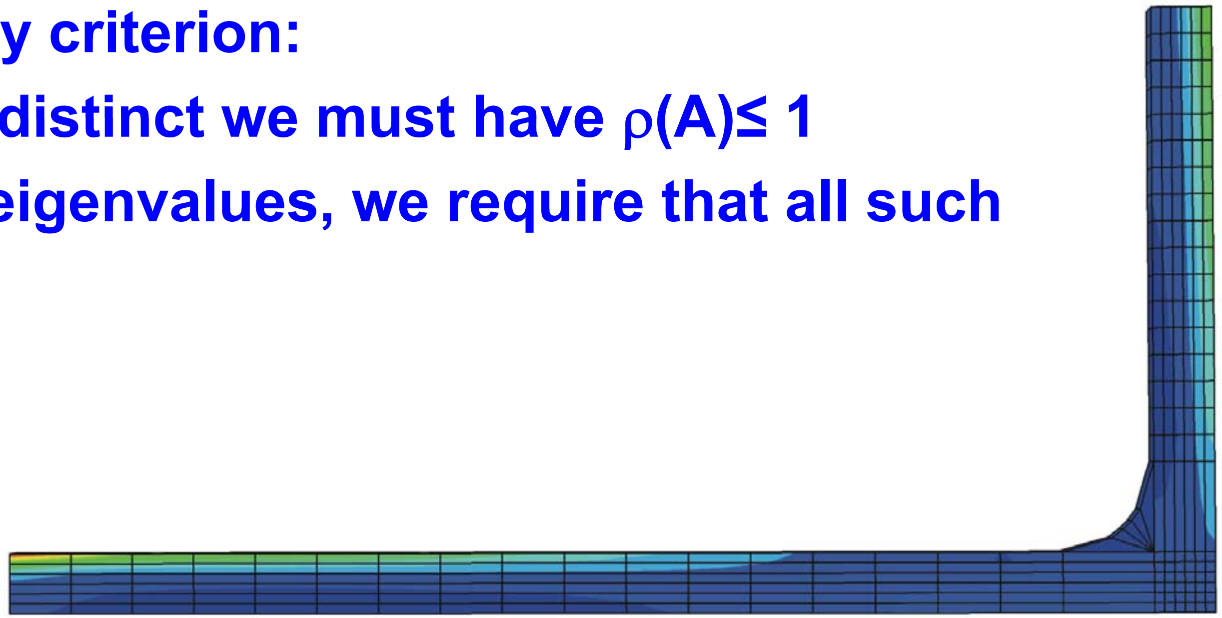
Stability analysis

- Now we can write $A^n=PJ^nP^{-1}$ and using this we can determine the stability of the time integration scheme

- If we now consider the spectral radius A defined as

$$\rho(\mathbf{A}) = \max_{i=1,2,\ldots} |\lambda_i|$$

- we can write the stability criterion:
- 1. If all eigenvalues are distinct we must have $\rho(A) \leq 1$
- 2. If A contain multiple eigenvalues, we require that all such must be smaller than 1



Stability analysis

- **Using simple Example (E 9.12) we can compare different integration methods using corresponding approximation operators**

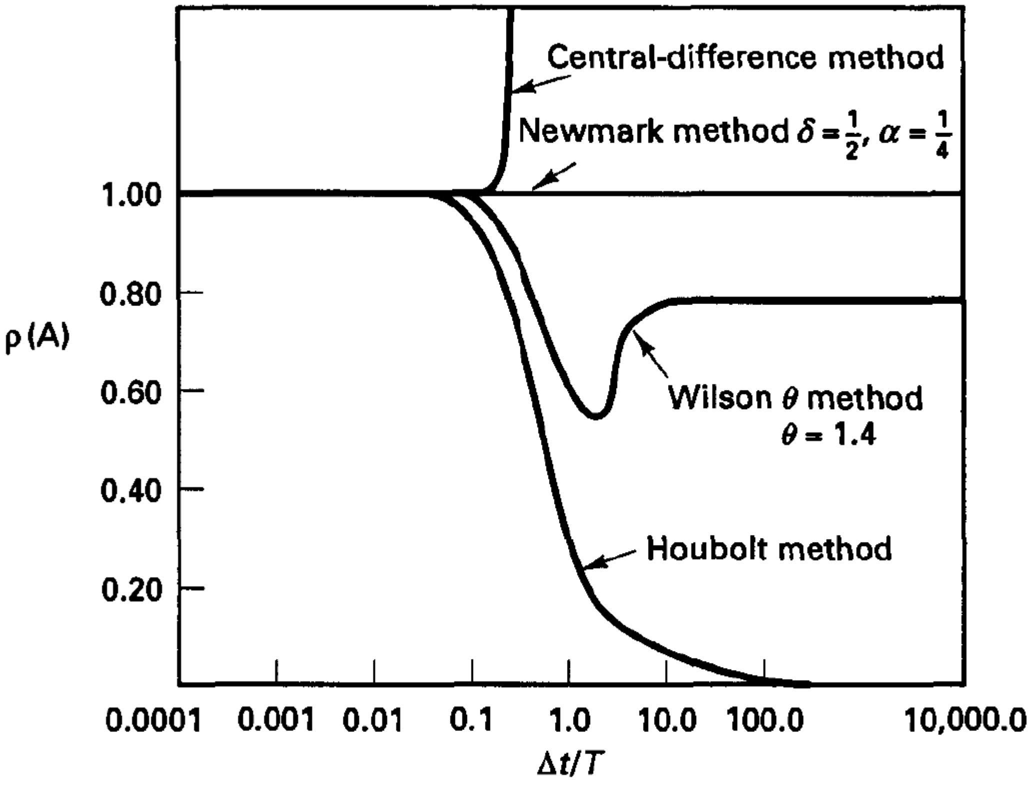
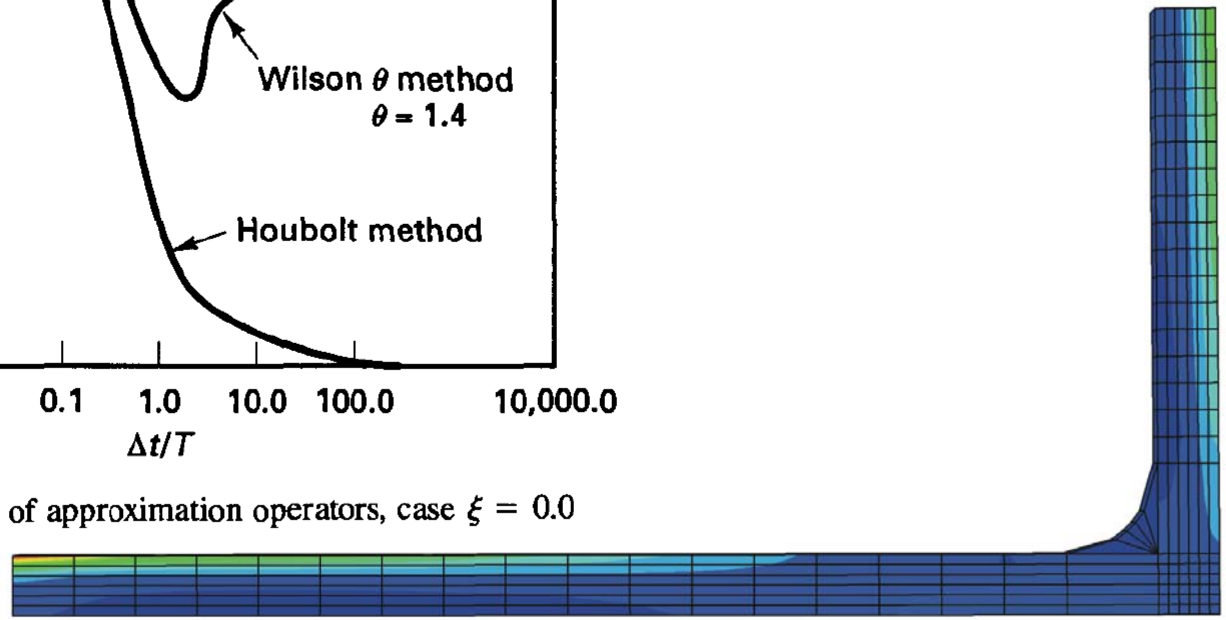
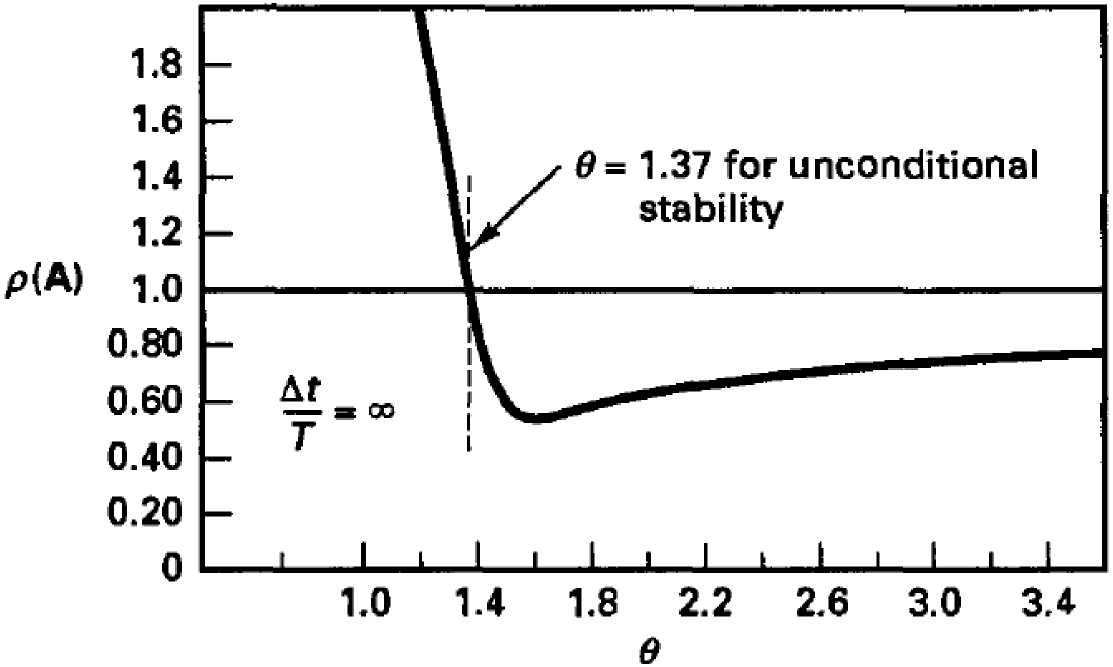


Figure 9.4 Spectral radii of approximation operators, case $\xi = 0.0$

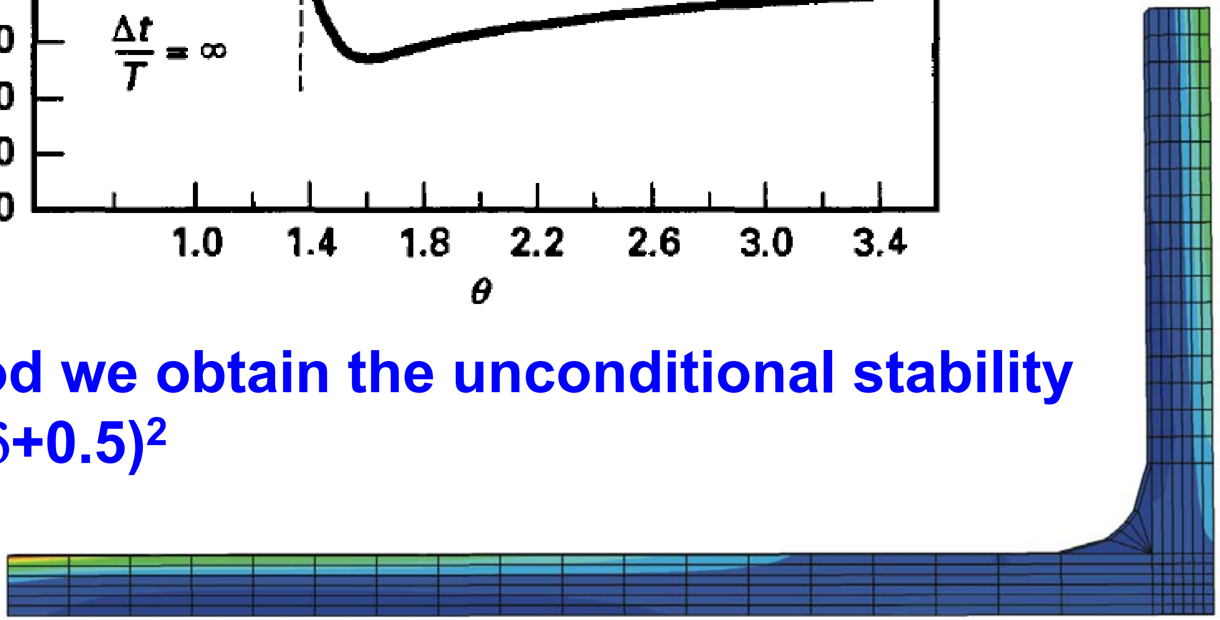


Stability analysis

- Furthermore, one can evaluate the optimum value of θ for the Wilson θ method



- For the Newmark method we obtain the unconditional stability for $\delta \geq 0.5$ and $\alpha \geq (0.25\delta+0.5)^2$

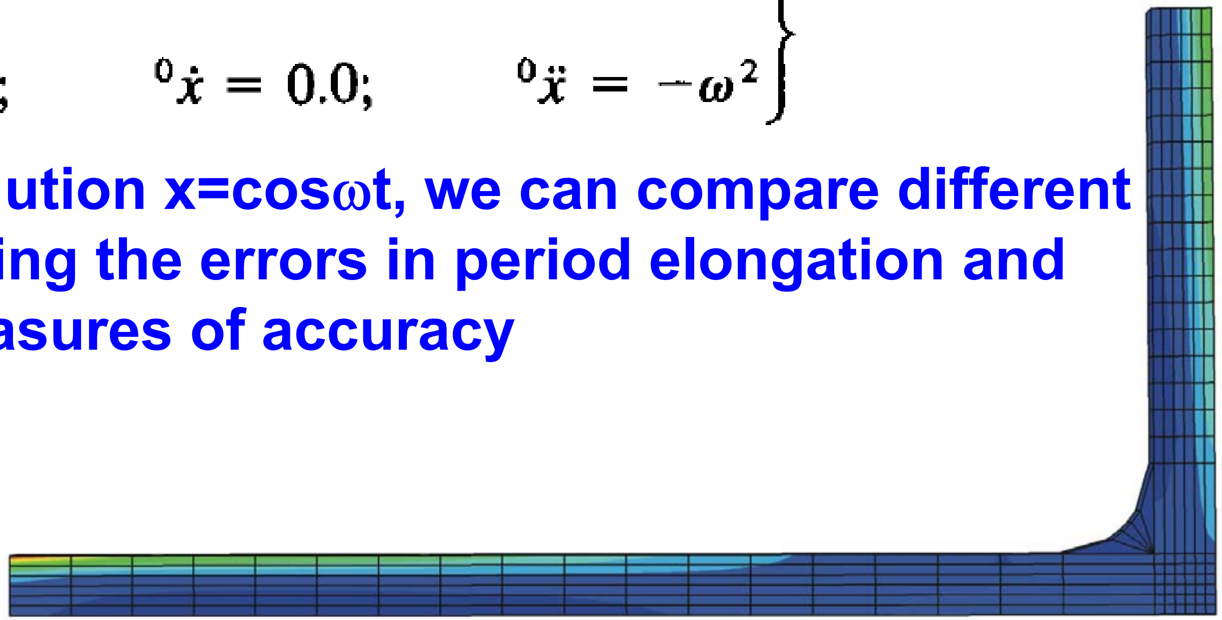


Accuracy analysis

- In general, using an unconditionally stable operator, the time step has to be chosen to yield an accurate and effective solution
- Integration accuracy can be assessed as a function of $\Delta t/T$, ξ and r, as we have seen before
- Considering the solution of the initial value problem defined by

$$\left.\begin{array}{c} \ddot{x} + \omega^2 x = 0 \\ {}^0x = 1.0; \qquad {}^0\dot{x} = 0.0; \qquad {}^0\ddot{x} = -\omega^2 \end{array}\right\}$$

- and having an exact solution $x = \cos\omega t$, we can compare different integration methods using the errors in period elongation and amplitude decay as measures of accuracy



Accuracy analysis

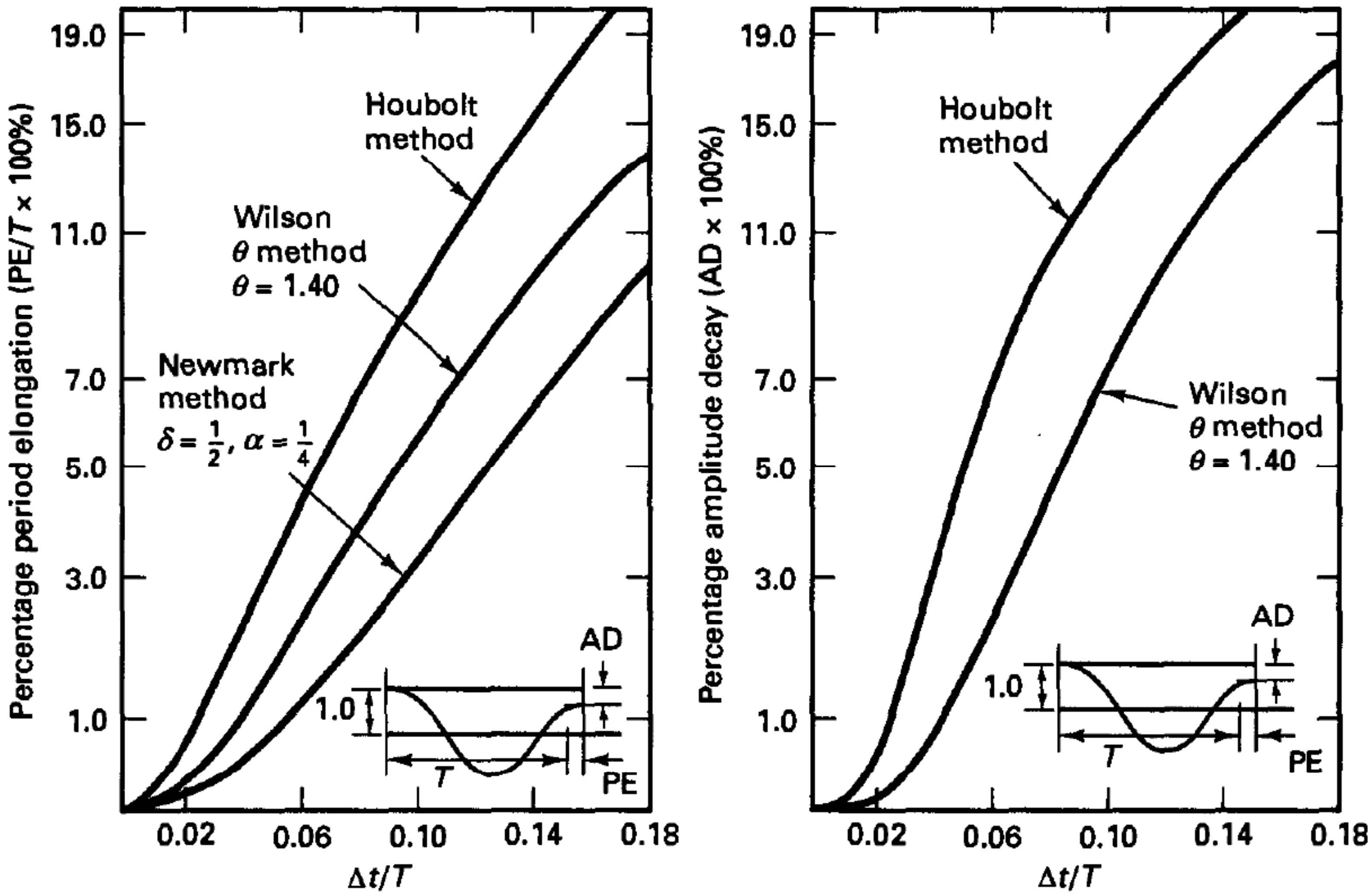
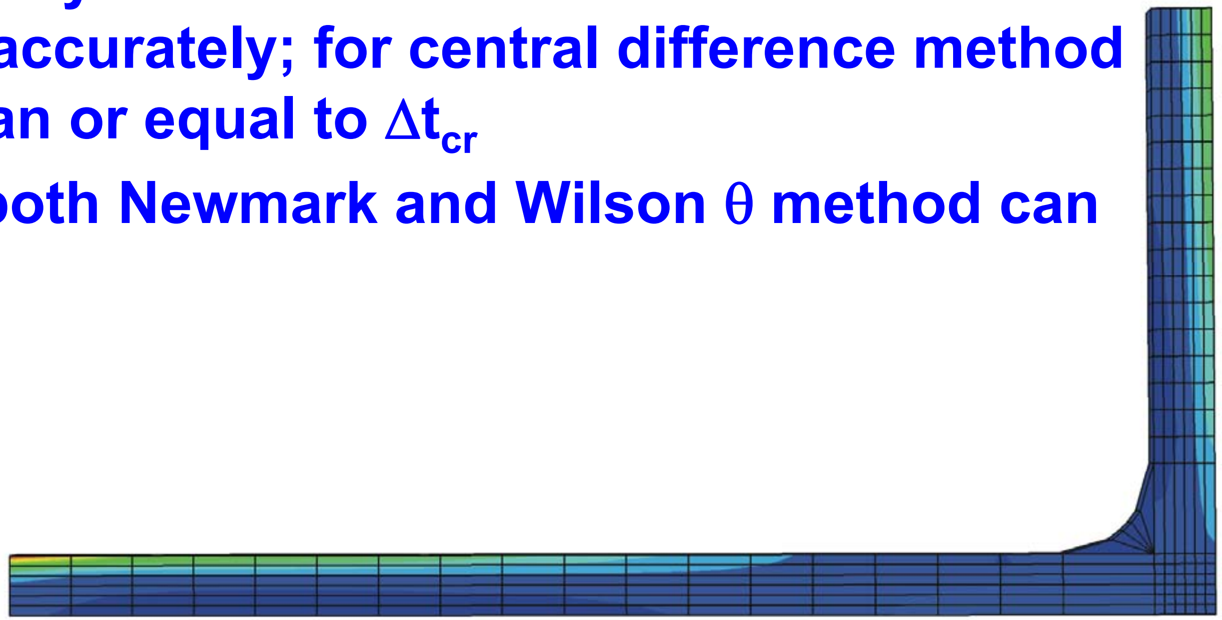


Figure 9.6 Percentage period elongations and amplitude decays

Accuracy analysis

- **All methods are accurate for very small ratios $\Delta t/T$ (0.01). For larger ratios and for given $\Delta t/T$ the Wilson θ method with θ=1.4 introduces less amplitude decay and period elongation than the Houbolt method, and the Newmark method the smallest percentage period elongation and no amplitude decay**
- **In sum, Δt has to be chosen small enough so that the response in all modes that significantly contribute to the total structure response is calculated accurately; for central difference method Δt should be smaller than or equal to Δt_{cr}**
- **For practical analysis, both Newmark and Wilson θ method can be used**



Some practical considerations

- **To obtain an effective solution of a dynamic response an appropriate time integration scheme must be chosen. This choice depends on the finite element idealization which in turn depends on a physical problem to be analysed (structural dynamics or a wave propagation problem)**
- **Structural dynamics: only the lowest modes are considered; time step Δt should equal to $T_{co}/20$, where $T_{co}=2\pi/\omega_{co}$ and $\omega_{co}=4\omega_u$, ω_u being highest frequency significantly contained in the loading**
- **Wave propagation: large number of frequencies are excited in the system; "effective length" of finite element should be $L_e=c\Delta t$, where $\Delta t=t_w/n$ and $t_w=L_w/c$. L_w is a wavelength, c wave speed and n number of time steps needed to represent the travel of the wave**