Mode Superposition — Example 9.7

1. Change of Basis to Modal Generalized Displacements

Equation of equilibrium :

$$M\ddot{U} + C\dot{U} + KU = R$$

Using the following transformation:

$$U(t) = PX(t)$$

P is a *n X n* square matrix.

X(t) time-dependent of order n.

Substituting,

$$\underbrace{P^T M P}_{\tilde{K}} \ddot{X}(t) + \underbrace{P^T C P}_{\tilde{C}} \dot{X}(t) + \underbrace{P^T K P}_{\tilde{P}} X(t) = \underbrace{P^T R}_{\tilde{R}}$$

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Equilibrium equations with damping neglected,

$$M\ddot{U} + KU = 0$$

The solution of this equation is:

$$U = \Phi \sin \omega (t - t_0)$$

ϕ vector of order n

t time variable

t_0 time constant

ω constant identified (frequency of vibration)

The second derivated:

$$\ddot{U} = -\omega^2 \Phi \sin \omega (t - t_0)$$

Substituing :

$$K\Phi = \omega^2 M\Phi$$

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We can write the n solutions like:

$$K\Phi = M\Phi\Omega^2$$

Eigenvectors are M-orthonormal:

$$\left.\begin{array}{c}\underbrace{\Phi^T K\Phi = \Omega^2}_{\tilde{K}} \\ \underbrace{\Phi^T M\Phi = I}_{\tilde{M}}\end{array}\right. \Rightarrow \quad P = \Phi \quad \Rightarrow \quad U(t) = \Phi X(t)$$

Equilibrium equation:

$$\ddot{X}(t) + \Phi^T C\Phi\dot{X}(t) + \Omega^2 X(t) = \Phi^T R(t)$$

2. Analysis with Damping Neglected

The velocity- dependent damping effects are not included:

$$\ddot{X}(t) + \Omega^2 X(t) = \Phi^T R(t)$$

Individual equations:

$$\left.\begin{aligned} \ddot{x}_i(t) + \omega_i^2 x_i(t) = r_i(t) \\ r_i(t) = \Phi_i^T R(t) \end{aligned}\right\} i = 1,2,\ldots,n. \qquad \Rightarrow \qquad \begin{aligned} x_i\big|_{t=0} &= \Phi_i^T M\,{}^0U \\ \dot{x}_i\big|_{t=0} &= \Phi_i^T M\,{}^0\dot{U} \end{aligned}$$

$$x_i\big|_{t=0} = \Phi_i^T M\,{}^0U$$

$$\dot{x}_i\big|_{t=0} = \Phi_i^T M\,{}^0\dot{U}$$

EXAMPLE 9.7: Use mode superposition to calculate the displacement response of the system considered in Exemples 9.1 to 9.4 and 9.6.

Consider a simple system: $M\ddot{U} + C\dot{U} + KU = R$

$$M = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{U}_1 \\ \ddot{U}_2 \end{bmatrix} + \begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \end{bmatrix} \quad \Rightarrow \quad K = \begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix}$$

$$R = \begin{bmatrix} 0 \\ 10 \end{bmatrix}$$

The generalized eigenproblem:

$$\begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix}\Phi = \varpi^2 \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\Phi$$

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Two solutions without derivation;

$$\omega_1^2 = 2; \quad \Phi_1 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$\omega_2^2 = 5; \quad \Phi_2 = \begin{bmatrix} \frac{1}{2}\sqrt{\frac{2}{3}} \\ -\sqrt{\frac{2}{3}} \end{bmatrix}$$

Equilibrium equation: $\ddot{X}(t) + \Phi^T C \Phi \dot{X}(t) + \Omega^2 X(t) = \Phi^T R(t)$

$$\ddot{X}(t) + \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} X(t) = \begin{bmatrix} \frac{10}{\sqrt{3}} \\ -10\sqrt{\frac{2}{3}} \end{bmatrix}$$

(1) Calculate the exact response by integrating each of two decoupled equilibrium equations exactly.

(2) Use the Newmark method with time step Δt=0.28 for the time integration.

Two equilibrium equations:

$$\ddot{x}_1 + 2x_1 = \frac{10}{\sqrt{3}} \quad \text{and} \quad \ddot{x}_2 + 5x_2 = -10\sqrt{\frac{2}{3}}$$

Initial conditions: $U\big|_{t=0} = 0$; $\dot{U}\big|_{t=0} = 0$

Using the equation: $\ddot{X}(t) + \Omega^2 X(t) = \Phi^T R(t)$

$$\left.\begin{aligned} \ddot{x}_i(t) + \omega_i^2 x_i(t) = r_i(t) \\ r_i(t) = \Phi_i^T R(t) \end{aligned}\right\} i = 1,2,\dots,n. \quad \Rightarrow \quad \begin{aligned} x_i\big|_{t=0} &= \Phi_i^T M\,{}^0U \\ \dot{x}_i\big|_{t=0} &= \Phi_i^T M\,{}^0\dot{U} \end{aligned}$$

$$x_1\big|_{t=0} = 0 \qquad \dot{x}_1\big|_{t=0} = 0$$

$$x_2\big|_{t=0} = 0 \qquad \dot{x}_2\big|_{t=0} = 0$$

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The exact solution is: $x_1 = \frac{5}{\sqrt{3}}(1-\cos\sqrt{2}t); \quad x_2 = 2\sqrt{\frac{2}{3}}(-1+\cos\sqrt{5}t)$

Using the relation: $U(t) = \Phi X(t)$

$$U(t) = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{2}\sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{3}} & -\sqrt{\frac{2}{3}} \end{bmatrix} X(t)$$

$$U(t) = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{2}\sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{3}} & -\sqrt{\frac{2}{3}} \end{bmatrix} \begin{bmatrix} \frac{5}{\sqrt{3}}(1-\cos\sqrt{2}t) \\ 2\sqrt{\frac{2}{3}}(-1+\cos\sqrt{5}t) \end{bmatrix}$$

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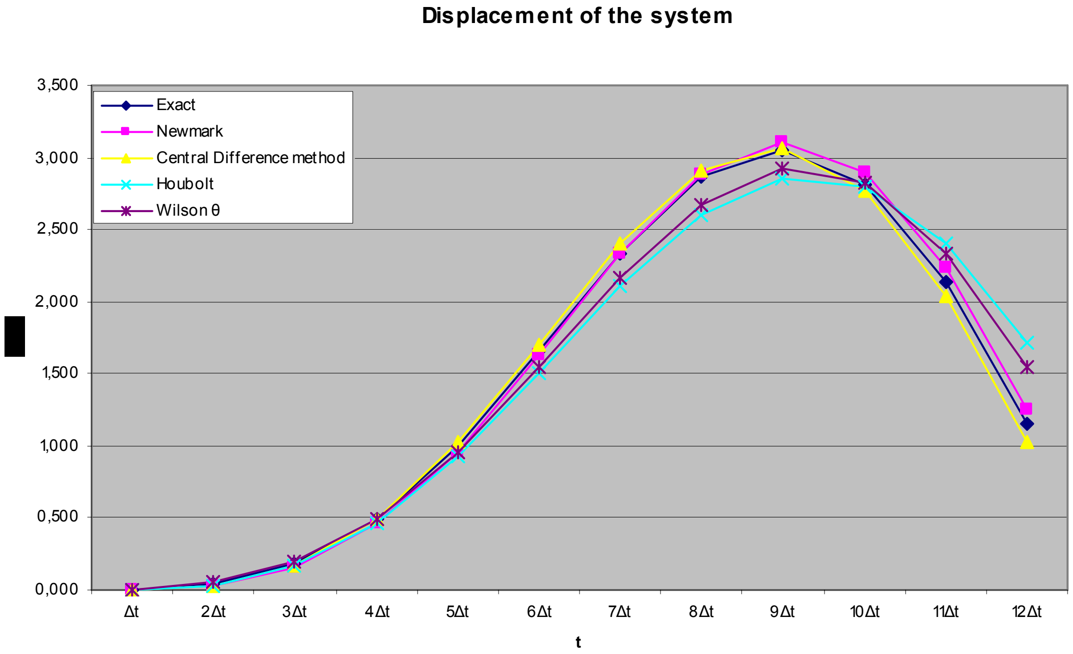
Employing Newmark method we obtain:

Time	Δt	2Δt	3Δt	4Δt	5Δt	6Δt	7Δt	8Δt	9Δt	10Δt	11Δt	12Δt
^{t}U	0.003	0.038	0.176	0.486	0.996	1.66	2.338	2.861	3.052	2.806	2.131	1.157
	0.382	1.41	2.78	4.09	5.00	5.29	4.986	4.277	3.457	2.806	2.484	2.489

Evaluating the desplacements, where Δt=0.28 we obtain:

Time	Δt	2Δt	3Δt	4Δt	5Δt	6Δt	7Δt	8Δt	9Δt	10Δt	11Δt	12Δt
X(t)	0.2258	0.8199	1.807	2.379	4.123	5.064	5.579	5.774	5.521	4.855	3.866	2.773
	-0.304	-0.792	-2.123	-2.939	-3.258	-2.632	4.986	-1.156	-0.330	-0.004	-0.248	-1.088


Displacement of the system
Exact
Newmark
Central Difference method
Houbolt
Wilson θ
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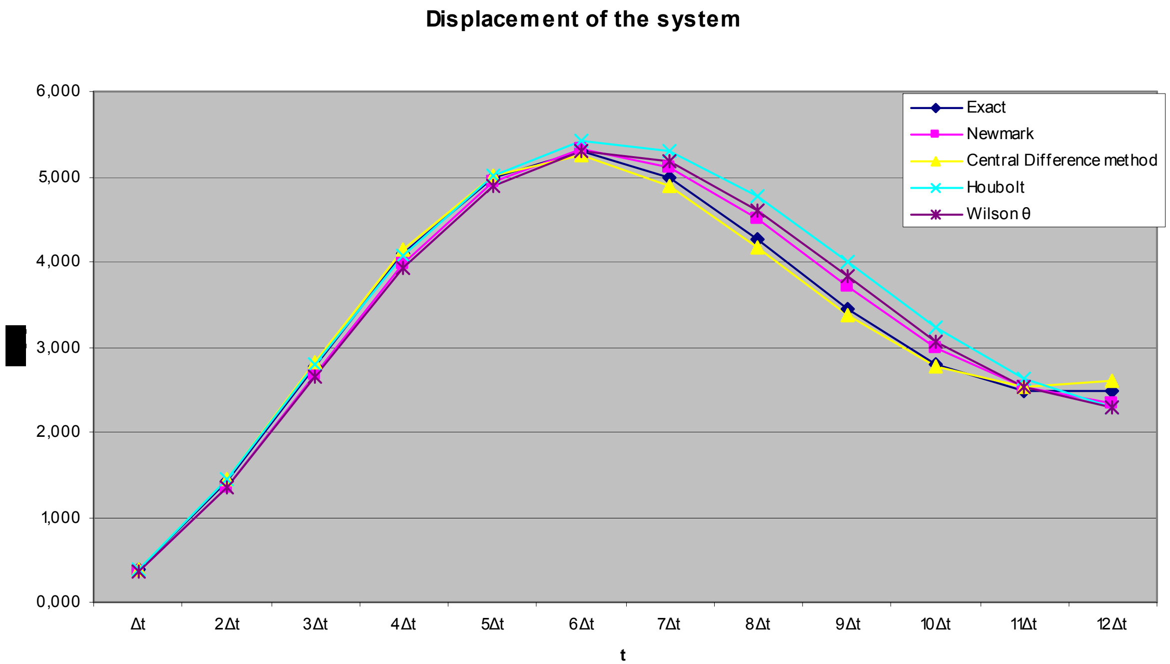
Displacement of the system
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