The Finite Element Method for the Analysis of Non-Linear and Dynamic Systems

Example 6.24 & 6.27
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EXAMPLE 6.24: Consider the four-node element shown in Fig. E6.24. The displacements of the element are given as a function of time.

Calculate the Cauchy stresses using the following two stress measures:

(i) Use the *total formulation* of the second Piola-Kirchhoff stress and Green-Lagrange strain tensors,

$$ {}_0^tS_{ij} = {}_0^tC_{ijrs}\,{}_0^t\epsilon_{rs} \qquad \text{(a)} $$

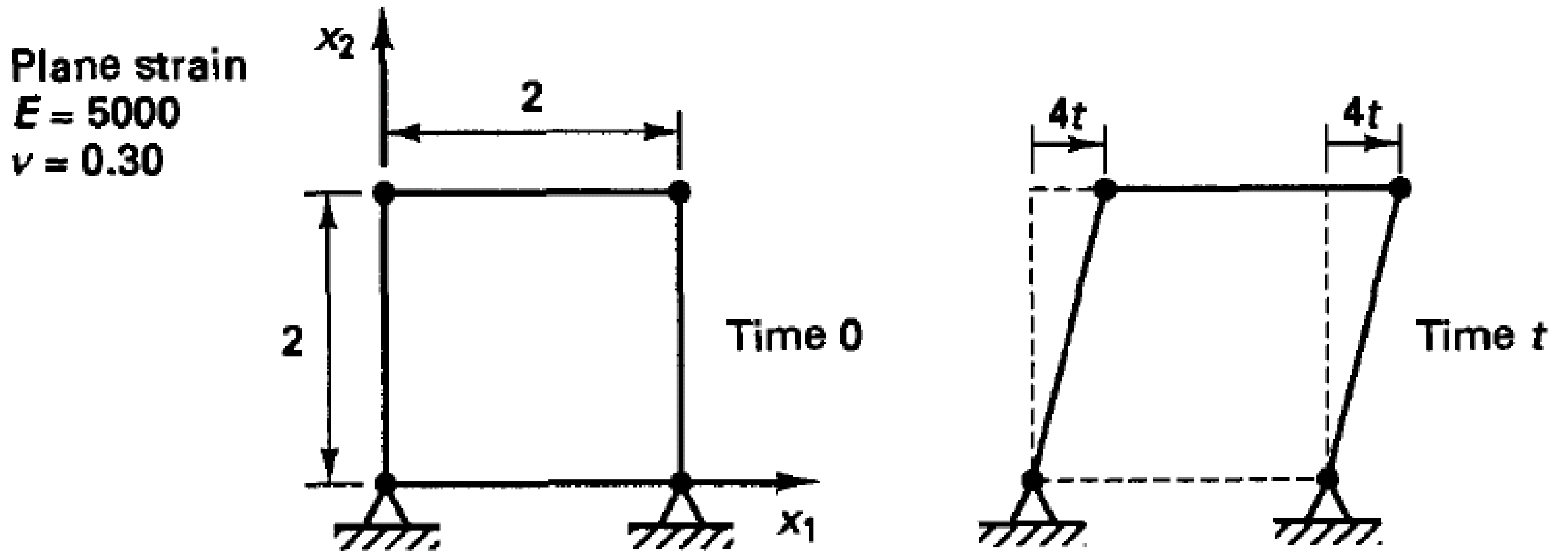

Figure E6.24 Four-node element subjected to motion

$$h_1 = \tfrac{1}{4}(1 + {}^0x_1)(1 + {}^0x_2); \qquad h_2 = \tfrac{1}{4}(1 - {}^0x_1)(1 + {}^0x_2)$$

$$h_3 = \tfrac{1}{4}(1 - {}^0x_1)(1 - {}^0x_2); \qquad h_4 = \tfrac{1}{4}(1 + {}^0x_1)(1 - {}^0x_2)$$

$$\frac{\partial h_1}{\partial {}^0x_1} = \frac{1}{4}(1 + {}^0x_2); \qquad \frac{\partial h_2}{\partial {}^0x_1} = -\frac{1}{4}(1 + {}^0x_2)$$

$$\frac{\partial h_3}{\partial {}^0x_1} = -\frac{1}{4}(1 - {}^0x_2); \qquad \frac{\partial h_4}{\partial {}^0x_1} = \frac{1}{4}(1 - {}^0x_2)$$

$$\frac{\partial h_1}{\partial {}^0x_2} = \frac{1}{4}(1 + {}^0x_1); \qquad \frac{\partial h_2}{\partial {}^0x_2} = \frac{1}{4}(1 - {}^0x_1)$$

$$\frac{\partial h_3}{\partial {}^0x_2} = -\frac{1}{4}(1 - {}^0x_1); \qquad \frac{\partial h_4}{\partial {}^0x_2} = -\frac{1}{4}(1 + {}^0x_1)$$

$${}^tx_i = \sum_{k=1}^{4} h_k \, {}^tx_i^k$$

$$\frac{\partial {}^tx_i}{\partial {}^0x_j} = \sum_{k=1}^{4} \left(\frac{\partial h_k}{\partial {}^0x_j}\right) {}^tx_i^k$$

Node coordinate at the time t

Node	$^t x_1$	$^t x_2$
1	1+4*t	1
2	4*t-1	1
3	-1	-1
4	1	-1

$$ {}_0^t X \equiv \begin{bmatrix} \dfrac{\partial\, ^t x_1}{\partial\, ^0 x_1} & \dfrac{\partial\, ^t x_1}{\partial\, ^0 x_2} \\ \dfrac{\partial\, ^t x_2}{\partial\, ^0 x_1} & \dfrac{\partial\, ^t x_2}{\partial\, ^0 x_2} \end{bmatrix} $$

The deformation gradient is

$$ {}_0^t\mathbf{X} = \begin{bmatrix} 1 & 2t \\ 0 & 1 \end{bmatrix} $$

$$ \begin{aligned} {}_0^t\boldsymbol{\epsilon} &= \tfrac{1}{2}({}_0^t\mathbf{U}\ {}_0^t\mathbf{R}^T\ {}_0^t\mathbf{R}\ {}_0^t\mathbf{U} - \mathbf{I}) \\ &= \boxed{\tfrac{1}{2}({}_0^t\mathbf{X}^T\ {}_0^t\mathbf{X} - \mathbf{I})} \\ &= \tfrac{1}{2}({}_0^t\mathbf{C} - \mathbf{I}) \end{aligned} \qquad (6.54) $$

$$ {}_0^t\boldsymbol{\epsilon} = \begin{bmatrix} 0 & t \\ t & 2t^2 \end{bmatrix} $$

$$ {}_0^t C_{ijrs} = \lambda \delta_{ij} \delta_{rs} + \mu(\delta_{ir}\delta_{js} + \delta_{is}\delta_{jr}) \qquad (6.185) $$

where λ and μ are the Lamé constants and δ_{ij} is the Kronecker delta,

$$ \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}; \qquad \mu = \frac{E}{2(1+\nu)} $$

$$ \delta_{ij} = \begin{cases} 0; & i \neq j \\ 1; & i = j \end{cases} $$

$C_{1111} = 6731$ $\qquad$ $C_{2222} = 6731$

$C_{1122} = C_{2211} = 2885$ $\qquad$ $C_{1212} = 1923$

$$ {}_0^t S_{ij} = {}_0^t C_{ijrs} \, {}_0^t \epsilon_{rs} $$

$$ {}_0^t S_{11} = 5770t^2; \qquad {}_0^t S_{22} = 13{,}462t^2; \qquad {}_0^t S_{12} = 3846t $$

$$ {}^{t}\boldsymbol{\tau} = \frac{{}^{t}\rho}{{}^{0}\rho}\, {}^{t}_{0}\mathbf{X}\, {}^{t}_{0}\mathbf{S}\, {}^{t}_{0}\mathbf{X}^{T} $$

$$ \begin{bmatrix} {}^{t}\tau_{11} \\ {}^{t}\tau_{22} \\ {}^{t}\tau_{12} \end{bmatrix} = \begin{bmatrix} 21{,}000t^{2} + 54{,}000t^{4} \\ 13{,}000t^{2} \\ 3800t + 27{,}000t^{3} \end{bmatrix} $$

(ii) Use the *rate formulation* of the Jaumann stress rate and the velocity strain tensors (see L. E. Malvern [A]),

$$ {}^{t}\overset{\circ}{\tau}_{ij} = {}_{t}C_{ijrs}\,{}^{t}D_{rs} \quad \text{(b)} $$

$$ \mathbf{L} = \dot{\mathbf{X}}\mathbf{X}^{-1} $$

velocity gradient tensor

$$ {}^{t}\mathbf{L} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}; $$

$\mathbf{L} = \mathbf{D} + \mathbf{W}$ **decomposition**

$$ \mathbf{D} = \frac{1}{2}\left(\mathbf{L} + \mathbf{L}^{T}\right) $$

$$ {}^{t}\mathbf{D} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; $$

Deformation rate tensor

$$ \mathbf{W} = \frac{1}{2}\left(\mathbf{L} - \mathbf{L}^{T}\right) $$

$$ {}^{t}\mathbf{W} = \begin{bmatrix} 0 & +1 \\ -1 & 0 \end{bmatrix} $$

Spin/rotation rate tensor

$$\begin{bmatrix} {}^t\overset{\nabla}{\tau}_{11} \\ {}^t\overset{\nabla}{\tau}_{22} \\ {}^t\overset{\nabla}{\tau}_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3846 \end{bmatrix}$$

We note that the components of the Jaumann stress rate tensor are given by (see L. E. Malvern [A])

$${}^t\overset{\nabla}{\tau}_{ij} = {}^t\dot{\tau}_{ij} + {}^t\tau_{ip}\,{}^tW_{pj} + {}^t\tau_{jp}\,{}^tW_{pi} \tag{c}$$

$$\begin{bmatrix} {}^t\dot{\tau}_{11} \\ {}^t\dot{\tau}_{22} \\ {}^t\dot{\tau}_{12} \end{bmatrix} = \begin{bmatrix} 2\,{}^t\tau_{12} \\ -2\,{}^t\tau_{12} \\ 3846 + {}^t\tau_{22} - {}^t\tau_{11} \end{bmatrix}$$

These differential equations can be solved to obtain (again to two significant figures and hence using $G = \dfrac{E}{2(1 + \nu)} \doteq 1900$)

$$\begin{bmatrix} {}^t\tau_{11} \\ {}^t\tau_{22} \\ {}^t\tau_{12} \end{bmatrix} = \begin{bmatrix} 1900(1 - \cos 2t) \\ -1900(1 - \cos 2t) \\ 1900 \sin 2t \end{bmatrix} \tag{d}$$

EXAMPLE 6.27: Consider element 2 in Example 4.5 and assume that in an elastoplastic analysis the stresses at time t in the element are such that the tangent moduli of the material are equal to $E/100$ for $0 \leq x \leq 40$ and equal to E for $40 < x \leq 80$ as illustrated in Fig. E6.27. Evaluate the tangent stiffness matrix ${}^t\mathbf{K}$ using one-, two-, three-, and four-point Gauss integration and compare these results with the exact stiffness matrix. Consider only material nonlinearities.

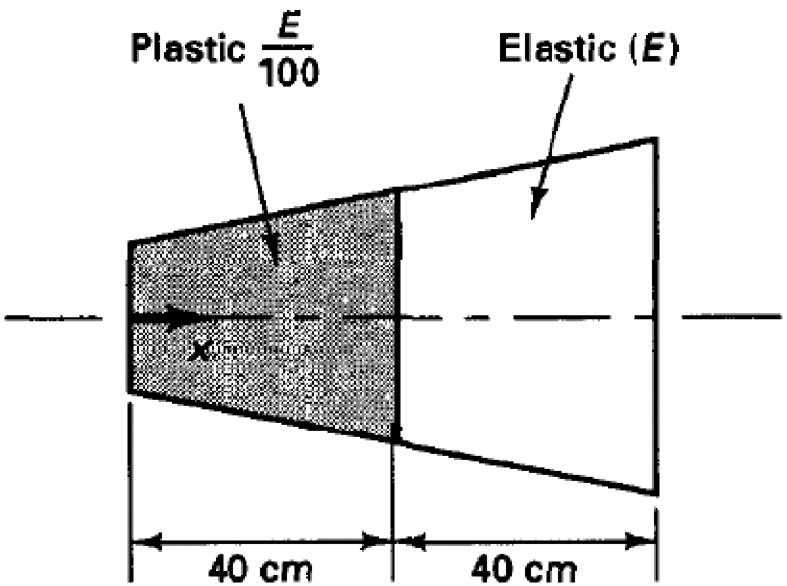

Figure E6.27 Element 2 of Example 4.5 in elastic-plastic conditions

$$\mathbf{H} = \begin{bmatrix} \left(1 - \frac{x}{80}\right) & \frac{x}{80} \end{bmatrix} \qquad C = E$$

$$\mathbf{B} = \begin{bmatrix} -\frac{1}{80} & \frac{1}{80} \end{bmatrix} \qquad A = \left(1 + \frac{x}{40}\right)^2 \text{ cm}^2$$

$$\mathbf{K} = \int_V \mathbf{B}^T \mathbf{C} \mathbf{B} \, dV \tag{5.27}$$

TABLE 5.6 *Sampling points and weights in Gauss-Legendre numerical integration (interval −1 to +1)*

n	r_i			α_i		
1	0.	(15 zeros)		2.	(15 zeros)	
2	±0.57735	02691	89626	1.00000	00000	00000
3	±0.77459	66692	41483	0.55555	55555	55556
	0.00000	00000	00000	0.88888	88888	88889
4	±0.86113	63115	94053	0.34785	48451	37454
	±0.33998	10435	84856	0.65214	51548	62546

One-point integration:

$$ {}^{t}\mathbf{K} = 2 \times 40\begin{bmatrix} -\frac{1}{80} \\ \frac{1}{80} \end{bmatrix}\frac{E}{100}\begin{bmatrix} -\frac{1}{80} & \frac{1}{80} \end{bmatrix}(1 + 1)^2 = 0.0005E\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} $$

Two-point integration:

$$ {}^{t}\mathbf{K} = 1 \times 40\begin{bmatrix} -\frac{1}{80} \\ \frac{1}{80} \end{bmatrix}\frac{E}{100}\begin{bmatrix} -\frac{1}{80} & \frac{1}{80} \end{bmatrix}\left(1 + 1 - \frac{1}{\sqrt{3}}\right)^2 $$

$$ + 1 \times 40\begin{bmatrix} -\frac{1}{80} \\ \frac{1}{80} \end{bmatrix}E\begin{bmatrix} -\frac{1}{80} & \frac{1}{80} \end{bmatrix}\left(1 + 1 + \frac{1}{\sqrt{3}}\right)^2 $$

$$ = 0.04164E\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} $$

Three-point integration:

$$'\mathbf{K} = \frac{5}{9}40\begin{bmatrix} -\frac{1}{80} \\ \frac{1}{80} \end{bmatrix}\frac{E}{100}\begin{bmatrix} -\frac{1}{80} & \frac{1}{80} \end{bmatrix}\left(1 + 1 - \frac{\sqrt{3}}{5}\right)^2$$

$$+ \frac{8}{9}40\begin{bmatrix} -\frac{1}{80} \\ \frac{1}{80} \end{bmatrix}\frac{E}{100}\begin{bmatrix} -\frac{1}{80} & \frac{1}{80} \end{bmatrix}(1 + 1)^2$$

$$+ \frac{5}{9}40\begin{bmatrix} -\frac{1}{80} \\ \frac{1}{80} \end{bmatrix}E\begin{bmatrix} -\frac{1}{80} & \frac{1}{80} \end{bmatrix}(1 + 1 + \sqrt{3}/5)^2$$

$$'\mathbf{K} = 0.02700E\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Four-point integration:

	r_i	α_i
$n = 4$	$\pm 0.8611\ldots$	$0.3478\ldots$
	$\pm 0.3399\ldots$	$0.6521\ldots$

$$'\mathbf{K} = 0.3478\ldots(40)\begin{bmatrix} -\frac{1}{80} \\ \frac{1}{80} \end{bmatrix}\frac{E}{100}\begin{bmatrix} -\frac{1}{80} & \frac{1}{80} \end{bmatrix}(1 + 1 - 0.8611\ldots)^2$$

$$+ \cdots$$

$$'\mathbf{K} = 0.04026E\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

The exact stiffness matrix is

$$
{}^{t}\mathbf{K} = \begin{bmatrix} -\frac{1}{80} \\ \frac{1}{80} \end{bmatrix} \frac{E}{100} \begin{bmatrix} -\frac{1}{80} & \frac{1}{80} \end{bmatrix} \left\{ \int_0^{40} \left(1 + \frac{y}{40}\right)^2 dy + \int_{40}^{80} 100\left(1 + \frac{y}{40}\right)^2 dy \right\}
$$

$$
= \begin{bmatrix} -\frac{1}{80} \\ \frac{1}{80} \end{bmatrix} E \begin{bmatrix} -\frac{1}{80} & \frac{1}{80} \end{bmatrix} \left\{ \frac{40}{300}\left(1 + \frac{y}{40}\right)^3 \Big|_0^{40} + \frac{40}{3}\left(1 + \frac{y}{40}\right)^3 \Big|_{40}^{80} \right\}
$$

$$
{}^{t}\mathbf{K} = 0.03973E \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}
$$