Method of Finite Elements II

Example 6.18 & 6.20

and necessary theory

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The incremental strains:

Principle of virtual work approximating by a Taylor Series

We get:

$$
\begin{aligned}
{}_0\epsilon_{11} &= {}_0u_{1,1} + {}^{t}_{0}u_{1,1}\,{}_0u_{1,1} + {}^{t}_{0}u_{2,1}\,{}_0u_{2,1} + \tfrac{1}{2}\left(({}_0u_{1,1})^2 + ({}_0u_{2,1})^2\right) \\
{}_0\epsilon_{22} &= {}_0u_{2,2} + {}^{t}_{0}u_{1,2}\,{}_0u_{1,2} + {}^{t}_{0}u_{2,2}\,{}_0u_{2,2} + \tfrac{1}{2}\left(({}_0u_{1,2})^2 + ({}_0u_{2,2})^2\right) \\
{}_0\epsilon_{12} &= \tfrac{1}{2}\left({}_0u_{1,2} + {}_0u_{2,1}\right) + \tfrac{1}{2}\left({}^{t}_{0}u_{1,1}\,{}_0u_{1,2} + {}^{t}_{0}u_{2,1}\,{}_0u_{2,2} + {}^{t}_{0}u_{1,2}\,{}_0u_{1,1} + {}^{t}_{0}u_{2,2}\,{}_0u_{2,1}\right) + \tfrac{1}{2}\left({}_0u_{1,1}\,{}_0u_{1,2} + {}_0u_{2,1}\,{}_0u_{2,2}\right)
\end{aligned}
$$

Keep in mind: $\boldsymbol{\varepsilon} = \boldsymbol{B}\hat{\boldsymbol{u}}$

where **û** is a vector with the element nodal displacements

We then get as the incremental strains:

$$ {}_0\epsilon_{11} = \underbrace{{}_0u_{1,1}}_{{}^t_0B_{L0}} + \underbrace{{}^t_0u_{1,1}\,{}_0u_{1,1} + {}^t_0u_{2,1}\,{}_0u_{2,1}}_{{}^t_0B_{L1}} + \underbrace{\tfrac{1}{2}\left(({}_0u_{1,1})^2 + ({}_0u_{2,1})^2\right)}_{{}^t_0B_{NL}} $$

$$ {}^t_0B_L := {}^t_0B_{L0} + {}^t_0B_{L1} $$

Motivation for introducing the B-Matrix:

Isoparametric elements

Displacements are interpolated in the same way as the geometry:

geometry → displacements

$$x = \sum_{i=1}^{q} h_i x_i \quad \rightarrow \quad u = \sum_{i=1}^{q} h_i u_i$$

where q is the number of nodes per element and h the interpolation functions

How to get the element strains?

The strains consist of the derivatives of the element displacements wrt to the local coordinates (constitutive equations). Whereas the element displacements are in natural coordinates.

the local coord‘s are of the form $x = f(r,s,t)$

the natural coord‘s are $r = f(x,y,z)$

Idea: relate these two by chain rule

Drawback:

to derive the partial derivatives wrt x, the explicit relations $r = f(x, y, z)$ would have to be known/evaluated, which are generally hard to compute.

Instead, applying chain rule to r:

$$\frac{\partial}{\partial r} = \frac{\partial}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial}{\partial z}\frac{\partial z}{\partial r}$$

$$\frac{\partial}{\partial r} = \frac{\partial}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial}{\partial z}\frac{\partial z}{\partial r}$$

Which leads us to the Jacobian Matrix

$$\underbrace{\begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial s} \\ \frac{\partial}{\partial t} \end{pmatrix}}_{\frac{\partial}{\partial \boldsymbol{r}}} = \underbrace{\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t} \end{pmatrix}}_{\boldsymbol{J}} \underbrace{\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}}_{\frac{\partial}{\partial \boldsymbol{x}}}$$

Which only useful in its inverted form:

$$\frac{\boldsymbol{\partial}}{\boldsymbol{\partial x}} = \boldsymbol{J}^{-1}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial r}}$$

Remember: the inverse of a regular 2x2-Matrix is

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{-1} = \frac{1}{det\boldsymbol{a}}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

The B-Matrices look the following:

$$
{}^t_0\mathbf{B}_{L0} = \begin{bmatrix}
{}_0h_{1,1} & 0 & {}_0h_{2,1} & 0 & {}_0h_{3,1} & 0 & \cdots & {}_0h_{N,1} & 0 \\
0 & {}_0h_{1,2} & 0 & {}_0h_{2,2} & 0 & {}_0h_{3,2} & \cdots & 0 & {}_0h_{N,2} \\
{}_0h_{1,2} & {}_0h_{1,1} & {}_0h_{2,2} & {}_0h_{2,1} & {}_0h_{3,2} & {}_0h_{3,1} & \cdots & {}_0h_{N,2} & {}_0h_{N,1} \\
\frac{h_1}{{}^0\bar{x}_1} & 0 & \frac{h_2}{{}^0\bar{x}_1} & 0 & \frac{h_3}{{}^0\bar{x}_1} & 0 & \cdots & \frac{h_N}{{}^0\bar{x}_1} & 0
\end{bmatrix}
$$

with

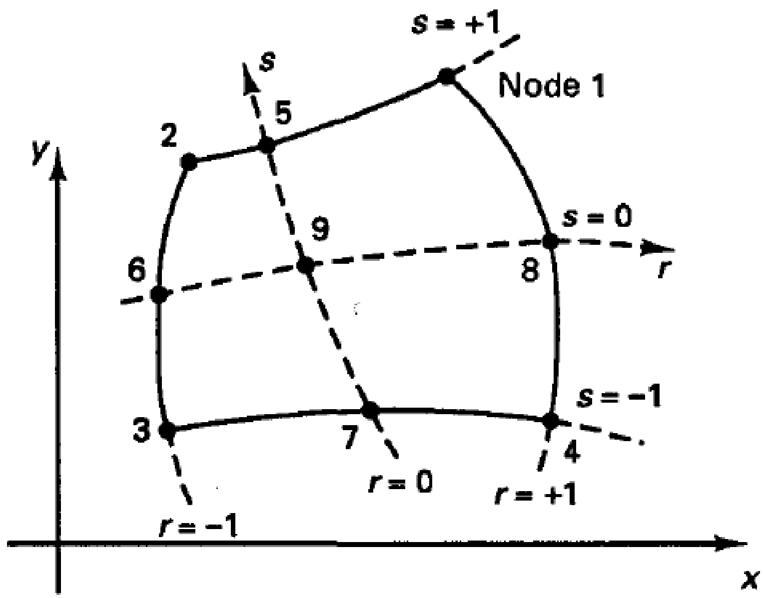
$$
{}_0h_{k,j} = \frac{\partial h_k}{\partial\, {}^0x_j}
$$

$$
{}_0^t\mathbf{B}_{L1} = \begin{bmatrix}
l_{11}\,{}_0h_{1,1} & l_{21}\,{}_0h_{1,1} & l_{11}\,{}_0h_{2,1} & l_{21}\,{}_0h_{2,1} & \cdots & l_{11}\,{}_0h_{N,1} & l_{21}\,{}_0h_{N,1} \\
l_{12}\,{}_0h_{1,2} & l_{22}\,{}_0h_{1,2} & l_{12}\,{}_0h_{2,2} & l_{22}\,{}_0h_{2,2} & \cdots & l_{12}\,{}_0h_{N,2} & l_{22}\,{}_0h_{N,2} \\
(l_{11}\,{}_0h_{1,2} + l_{12}\,{}_0h_{1,1}) & (l_{21}\,{}_0h_{1,2} + l_{22}\,{}_0h_{1,1}) & (l_{11}\,{}_0h_{2,2} + l_{12}\,{}_0h_{2,1}) & (l_{21}\,{}_0h_{2,2} + l_{22}\,{}_0h_{2,1}) & \cdots & (l_{11}\,{}_0h_{N,2} + l_{12}\,{}_0h_{N,1}) & (l_{21}\,{}_0h_{N,2} + l_{22}\,{}_0h_{N,1}) \\
l_{33}\,\frac{h_1}{{}^0\bar{x}_1} & 0 & l_{33}\,\frac{h_2}{{}^0\bar{x}_1} & 0 & \cdots & l_{33}\,\frac{h_N}{{}^0\bar{x}_1} & 0
\end{bmatrix}
$$

with $$l_{ij} = \sum_{k=1}^{N} {}_0h_{k,j}\,{}^tu_i^k$$

$$
{}_0^t\mathbf{B}_{NL} = \begin{bmatrix}
{}_0h_{1,1} & 0 & {}_0h_{2,1} & 0 & {}_0h_{3,1} & 0 & \cdots & {}_0h_{N,1} & 0 \\
{}_0h_{1,2} & 0 & {}_0h_{2,2} & 0 & {}_0h_{3,2} & 0 & \cdots & {}_0h_{N,2} & 0 \\
0 & {}_0h_{1,1} & 0 & {}_0h_{2,1} & 0 & {}_0h_{3,1} & \cdots & 0 & {}_0h_{N,1} \\
0 & {}_0h_{1,2} & 0 & {}_0h_{2,2} & 0 & {}_0h_{3,2} & \cdots & 0 & {}_0h_{N,2} \\
\frac{h_1}{{}^0\bar{x}_1} & 0 & \frac{h_2}{{}^0\bar{x}_1} & 0 & \frac{h_3}{{}^0\bar{x}_1} & 0 & \cdots & \frac{h_N}{{}^0\bar{x}_1} & 0
\end{bmatrix}
$$

the interpolation functions for 2d-elements with 4 nodes:



$$h_1 = \frac{1}{4}(1+r)(1+s)$$

$$h_2 = \frac{1}{4}(1-r)(1+s)$$

$$h_3 = \frac{1}{4}(1-r)(1-s)$$

$$h_4 = \frac{1}{4}(1+r)(1-s)$$

EXAMPLE 6.18: Establish the matrices ${}^t_0\mathbf{B}_{L0}$, ${}^t_0\mathbf{B}_{L1}$, and ${}^t_0\mathbf{B}_{NL}$ corresponding to the TL formulation for the two-dimensional plane strain element shown in Fig. E6.18.

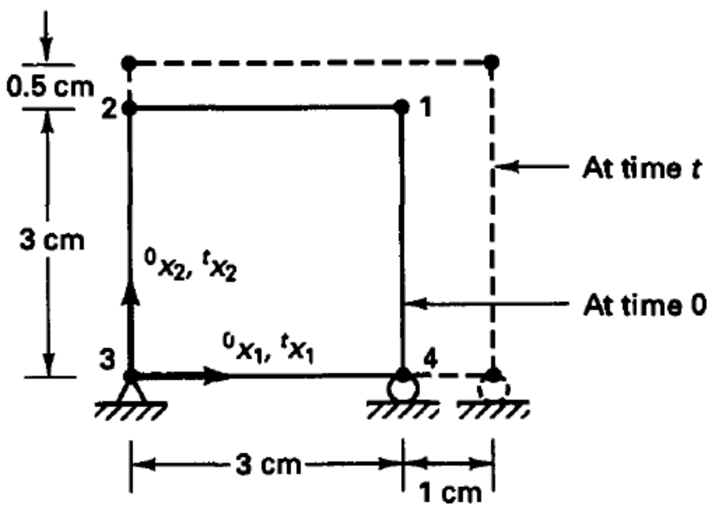


Figure E6.18 Four-node plane strain element in large displacement/large strain conditions

$$ {}^tu_1^1 = 1; \qquad {}^tu_2^1 = 0.5 $$
$$ {}^tu_1^2 = 0; \qquad {}^tu_2^2 = 0.5 $$
$$ {}^tu_1^3 = 0; \qquad {}^tu_2^3 = 0 $$
$$ {}^tu_1^4 = 1; \qquad {}^tu_2^4 = 0 $$

Isoparametric:

$$
\begin{aligned}
{}^{0}x_1 &= h_1\ {}^{0}x_1^1 + h_2\ {}^{0}x_1^2 + h_3\ {}^{0}x_1^3 + h_4\ {}^{0}x_1^4 \\
&= \frac{1}{4}(3x_1 + 6r + 6)
\end{aligned}
$$

Jacobian element: ${}^{0}J_{11} = \dfrac{\partial\ {}^{0}x_1}{\partial r} = \dfrac{3}{2}$

The other elements are derived analogous. This leads to:

$$
{}^{0}\mathbf{J} = \begin{bmatrix} \frac{3}{2} & 0 \\ 0 & \frac{3}{2} \end{bmatrix}
$$

For the B-Matrices, we need $\;{}_0h_{k,j} = \dfrac{\partial h_k}{\partial\, {}^0x_j}$ $\;\rightarrow\;$ $\dfrac{\partial}{\partial x} = J^{-1}\dfrac{\partial}{\partial r}$

$$\begin{pmatrix} \dfrac{\partial}{\partial x_1} \\ \dfrac{\partial}{\partial x_2} \end{pmatrix} = \begin{pmatrix} {}^2/_3 & 0 \\ 0 & {}^2/_3 \end{pmatrix} \begin{pmatrix} \dfrac{\partial}{\partial r} \\ \dfrac{\partial}{\partial s} \end{pmatrix}$$

Which allows us to write:

$$\frac{\partial h_1}{\partial\, {}^0x_1} = \frac{2}{3}\frac{\partial h_1}{\partial r} = \frac{2}{3}\frac{\partial \frac{1}{4}(1+r+s+rs)}{\partial r} = \frac{1}{6}(1+s)$$

This is how we assemble the ${}_0^tB_{L0}$ -Matrix:

$${}_0^t\mathbf{B}_{L0} =$$

$$\frac{1}{6}\begin{bmatrix} (1+s) & 0 & -(1+s) & 0 & -(1-s) & 0 & (1-s) & 0 \\ 0 & (1+r) & 0 & (1-r) & 0 & -(1-r) & 0 & -(1+r) \\ (1+r) & (1+s) & (1-r) & -(1+s) & -(1-r) & -(1-s) & -(1+r) & (1-s) \end{bmatrix}$$

For ${}_0^tB_{L1}$ we additionally need:

$${}_0^tu_{i,j} = l_{ij} = \sum_{k=1}^{N} {}_0h_{k,j} \;\; {}^tu_i^k$$

For the only nonzero l's, this is:

$$l_{11} = \sum_{k=1}^{4} {}_0h_{k,1} \, {}^tu_1^k = \tfrac{2}{3}\{h_{1,r} \, {}^tu_1^1 + h_{4,r} \, {}^tu_1^4\} = \tfrac{1}{3}$$

$$l_{22} = \sum_{k=1}^{4} {}_0h_{k,2} \, {}^tu_2^k = \tfrac{2}{3}\{h_{1,s} \, {}^tu_2^1 + h_{2,s} \, {}^tu_2^2\} = \tfrac{1}{6}$$

Premultiplying ${}_0^tB_{L0}$ with these l's gives us ${}_0^tB_{L1}$

$$
{}_0^t\mathbf{B}_{L1} = \frac{1}{36}\begin{bmatrix} 2(1+s) & 0 & -2(1+s) & 0 & -2(1-s) & 0 & 2(1-s) & 0 \\ 0 & (1+r) & 0 & (1-r) & 0 & -(1-r) & 0 & -(1+r) \\ 2(1+r) & (1+s) & 2(1-r) & -(1+s) & -2(1-r) & -(1-s) & -2(1+r) & (1-s) \end{bmatrix}
$$

Getting ${}_{0}^{t}B_{NL}$ is straightforward. We just insert the previously calculated terms:

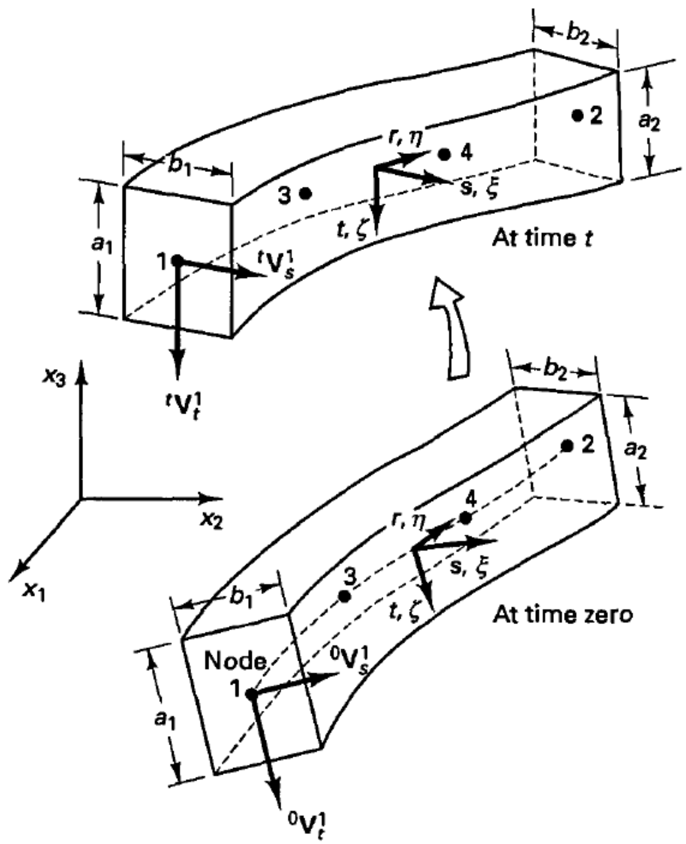
$$
{}_{0}^{t}\mathbf{B}_{NL} = \frac{1}{6}\begin{bmatrix} (1+s) & 0 & -(1+s) & 0 & -(1-s) & 0 & (1-s) & 0 \\ (1+r) & 0 & (1-r) & 0 & -(1-r) & 0 & -(1+r) & 0 \\ 0 & (1+s) & 0 & -(1+s) & 0 & -(1-s) & 0 & (1-s) \\ 0 & (1+r) & 0 & (1-r) & 0 & -(1-r) & 0 & -(1+r) \end{bmatrix}
$$

That‘s all of the magic for Example 6.18!

The geometry of a beam element at time t

$$ {}^{t}x_i = \sum_{k=1}^{q} h_k\, {}^{t}x_i^k + \frac{t}{2}\sum_{k=1}^{q} a_k h_k\, {}^{t}V_{ti}^k + \frac{s}{2}\sum_{k=1}^{q} b_k h_k\, {}^{t}V_{si}^k \qquad i = 1, 2, 3 $$

i…	Dimensions
q…	Number of nodes
a_k, b_k…	Cross-sections
h_k…	Interpolation functions
V…	See next slides



With the displacements components being

$$ {}^{t}u_i = {}^{t}x_i - {}^{0}x_i $$

$$ u_i = {}^{t+\Delta t}x_i - {}^{t}x_i $$

Inserting the previous equation in these two yields

$$ {}^{t}u_i = \sum_{k=1}^{q} h_k \, {}^{t}u_i^k + \frac{t}{2} \sum_{k=1}^{q} a_k h_k ({}^{t}V_{ti}^k - {}^{0}V_{ti}^k) + \frac{s}{2} \sum_{k=1}^{q} b_k h_k ({}^{t}V_{si}^k - {}^{0}V_{si}^k) $$

$$ u_i = \sum_{k=1}^{q} h_k u_i^k + \frac{t}{2} \sum_{k=1}^{q} a_k h_k V_{ti}^k + \frac{s}{2} \sum_{k=1}^{q} b_k h_k V_{si}^k $$

Where $V_{ti}^k = {}^{t+\Delta t}V_{ti}^k - {}^tV_{ti}^k$

$$V_{si}^k = {}^{t+\Delta t}V_{si}^k - {}^tV_{si}^k$$

are the nodal rotations, approximated by a 2nd order Taylor Series:

$$\mathbf{V}_t^k = \boldsymbol{\theta}_k \times {}^t\mathbf{V}_t^k + \frac{1}{2}\boldsymbol{\theta}_k \times (\boldsymbol{\theta}_k \times {}^t\mathbf{V}_t^k)$$

$$\mathbf{V}_s^k = \boldsymbol{\theta}_k \times {}^t\mathbf{V}_s^k + \frac{1}{2}\boldsymbol{\theta}_k \times (\boldsymbol{\theta}_k \times {}^t\mathbf{V}_s^k)$$

$\boldsymbol{\theta}_k$ are the rotational DOF. (will become clear in the example)

EXAMPLE 6.20: Consider the two-node beam element shown in Fig. E6.20. Evaluate the coordinate and displacement interpolations and derivatives that are required for the calculation of the strain-displacement matrices of the UL and TL formulations.

$$ {}^tV_s^1 = \begin{pmatrix} -\sin\ {}^t\theta_1 \\ \cos\ {}^t\theta_1 \end{pmatrix} $$

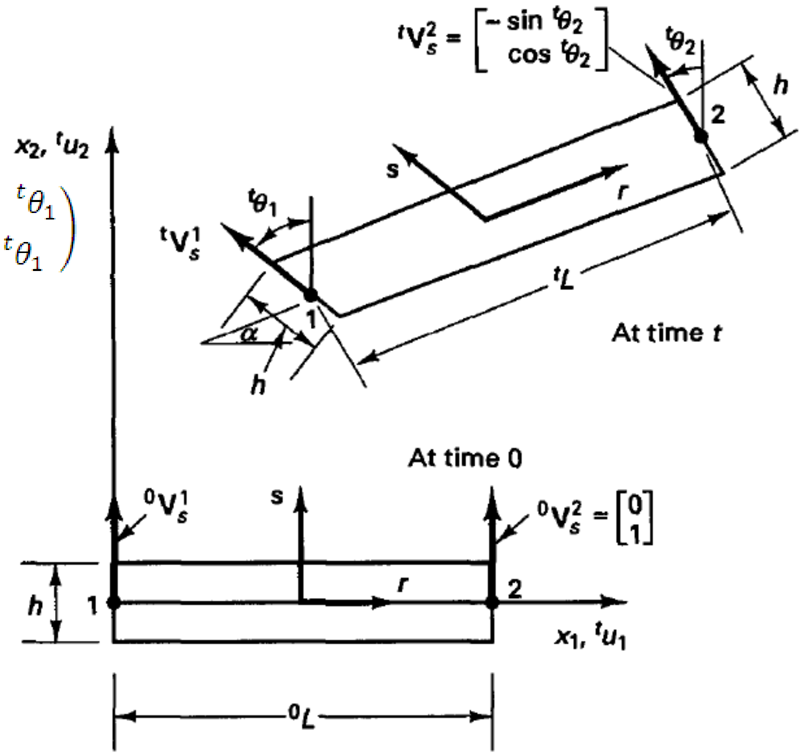


Figure E6.20 Two-node beam element in large displacements and rotations

Using the beam geometry equation:

$$^{t}x_1 = \left(\frac{1-r}{2}\right){}^{t}x_1^1 + \left(\frac{1+r}{2}\right){}^{t}x_1^2 - \frac{sh}{2}\left(\frac{1-r}{2}\right)\sin {}^{t}\theta_1 - \frac{sh}{2}\left(\frac{1+r}{2}\right)\sin {}^{t}\theta_2$$

$$^{t}x_2 = \left(\frac{1-r}{2}\right){}^{t}x_2^1 + \left(\frac{1+r}{2}\right){}^{t}x_2^2 + \frac{sh}{2}\left(\frac{1-r}{2}\right)\cos {}^{t}\theta_1 + \frac{sh}{2}\left(\frac{1+r}{2}\right)\cos {}^{t}\theta_2$$

h_1 h_2 h_1 h_2

Corresponds to a_k, b_k

1st comp of ${}^{t}V_s^{1}$

Setting $^{0}x_{1}^{1} = 0$ (Taking the beam back to reference configuration)

$$^{0}x_{1}^{2} = {}^{0}L$$

$$\theta_i = 0$$

Gives

$$^{0}x_{1} = \left(\frac{1+r}{2}\right) {}^{0}L$$

$$^{0}x_{2} = \frac{sh}{2}$$

Applying the above equations to

$$ {}^t u_i = {}^t x_i - {}^0 x_i $$

$$ u_i = {}^{t+\Delta t} x_i - {}^t x_i $$

leads to

$$ {}^t u_1 = \left(\frac{{}^t x_1^1 + {}^t x_1^2 - {}^0 L}{2}\right) + \left(\frac{{}^t x_1^2 - {}^t x_1^1 - {}^0 L}{2}\right) r - \frac{sh}{2}\left[\left(\frac{1-r}{2}\right)\sin {}^t\theta_1 + \left(\frac{1+r}{2}\right)\sin {}^t\theta_2\right] $$

$$ {}^t u_2 = \left(\frac{{}^t x_2^1 + {}^t x_2^2}{2}\right) + \left(\frac{{}^t x_2^2 - {}^t x_2^1}{2}\right) r + \frac{sh}{2}\left[\left(\frac{1-r}{2}\right)\cos {}^t\theta_1 + \left(\frac{1+r}{2}\right)\cos {}^t\theta_2 - 1\right] $$

$$ u_1 = \frac{1-r}{2} u_1^1 + \frac{1+r}{2} u_1^2 + \frac{sh}{2}\left(\frac{1-r}{2}\right)\left[(-\cos {}^t\theta_1)\theta_1 + \frac{1}{2}\sin {}^t\theta_1(\theta_1)^2\right] + \frac{sh}{2}\left(\frac{1+r}{2}\right)\left[(-\cos {}^t\theta_2)\theta_2 + \frac{1}{2}\sin {}^t\theta_2(\theta_2)^2\right] $$

$$ u_2 = \frac{1-r}{2} u_2^1 + \frac{1+r}{2} u_2^2 + \frac{sh}{2}\left(\frac{1-r}{2}\right)\left[(-\sin {}^t\theta_1)\theta_1 - \frac{1}{2}\cos {}^t\theta_1(\theta_1)^2\right] + \frac{sh}{2}\left(\frac{1+r}{2}\right)\left[(-\sin {}^t\theta_2)\theta_2 - \frac{1}{2}\cos {}^t\theta_2(\theta_2)^2\right] $$

How do we get these terms?

$$\mathbf{V}_t^k = \boldsymbol{\theta}_k \times {}^t\mathbf{V}_t^k + \frac{1}{2}\boldsymbol{\theta}_k \times (\boldsymbol{\theta}_k \times {}^t\mathbf{V}_t^k)$$

$$\mathbf{V}_s^k = \boldsymbol{\theta}_k \times {}^t\mathbf{V}_s^k + \frac{1}{2}\boldsymbol{\theta}_k \times (\boldsymbol{\theta}_k \times {}^t\mathbf{V}_s^k)$$

With $\boldsymbol{\theta}_1 = \begin{pmatrix} 0 \\ 0 \\ \theta_1 \end{pmatrix}, \boldsymbol{\theta}_2 = \begin{pmatrix} 0 \\ 0 \\ \theta_2 \end{pmatrix}$

Showing these calculations representatively for node 1…

$$ {}^{t}V_s^{1} = \begin{pmatrix} 0 \\ 0 \\ {}^{t}\theta_1 \end{pmatrix} \times \begin{pmatrix} -\sin\ {}^{t}\theta_1 \\ \cos\ {}^{t}\theta_1 \\ 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 \\ 0 \\ {}^{t}\theta_1 \end{pmatrix} \times \left(\begin{pmatrix} 0 \\ 0 \\ {}^{t}\theta_1 \end{pmatrix} \times \begin{pmatrix} -\sin\ {}^{t}\theta_1 \\ \cos\ {}^{t}\theta_1 \\ 0 \end{pmatrix} \right) $$

$$ \longrightarrow \quad {}^{t}V_s^{1} = \begin{pmatrix} -\ {}^{t}\theta_1 \cos\ {}^{t}\theta_1 \\ -\ {}^{t}\theta_1 \sin\ {}^{t}\theta_1 \\ 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} -\left({}^{t}\theta_1\right)^2 \sin\ {}^{t}\theta_1 \\ -\left({}^{t}\theta_1\right)^2 \cos\ {}^{t}\theta_1 \\ 0 \end{pmatrix} $$

${}^{t}V_s^{2}$ is derived in the same manner with ${}^{t}\boldsymbol{\theta}_2 = \begin{pmatrix} 0 \\ 0 \\ {}^{t}\theta_2 \end{pmatrix}$

For the UL formulation, we require the Jacobian, therefore

$$\frac{\partial\,{}^t x_1}{\partial r} = \frac{L\cos\alpha}{2} - \frac{sh}{4}(\sin{}^t\theta_2 - \sin{}^t\theta_1)$$

$$\frac{\partial\,{}^t x_1}{\partial s} = \left(-\frac{h}{2}\right)\left[\left(\frac{1-r}{2}\right)\sin{}^t\theta_1 + \left(\frac{1+r}{2}\right)\sin{}^t\theta_2\right]$$

$$\frac{\partial\,{}^t x_2}{\partial r} = \frac{L\sin\alpha}{2} + \frac{sh}{4}(\cos{}^t\theta_2 - \cos{}^t\theta_1)$$

$$\frac{\partial\,{}^t x_2}{\partial s} = \frac{h}{2}\left[\left(\frac{1-r}{2}\right)\cos{}^t\theta_1 + \left(\frac{1+r}{2}\right)\cos{}^t\theta_2\right]$$

simply by taking the derivatives of the beam geometry equations and assuming $L = {}^0L = {}^tL$

For the TL formulation we need the Jacobian, too

$$ {}^{0}\mathbf{J} = \begin{bmatrix} \dfrac{{}^{0}L}{2} & 0 \\ 0 & \dfrac{h}{2} \end{bmatrix} $$

this we get by taking the derivatives from

$$ {}^{0}x_1 = \left(\frac{1 + r}{2}\right) {}^{0}L $$

$$ {}^{0}x_2 = \frac{sh}{2} $$

Finally, the initial displacements are, by using the isoparametric formulation, i.e. the inverse of the Jacobian

$$ {}_0^t u_{1,1} = (\cos\alpha - 1) - \frac{sh}{2L}(\sin {}^t\theta_2 - \sin {}^t\theta_1) $$

$$ {}_0^t u_{1,2} = -\left(\frac{1-r}{2}\right)\sin {}^t\theta_1 - \left(\frac{1+r}{2}\right)\sin {}^t\theta_2 $$

$$ {}_0^t u_{2,1} = \sin\alpha + \frac{sh}{2L}(\cos {}^t\theta_2 - \cos {}^t\theta_1) $$

$$ {}_0^t u_{2,2} = \left(\frac{1-r}{2}\right)\cos {}^t\theta_1 + \left(\frac{1+r}{2}\right)\cos {}^t\theta_2 - 1 $$

again assuming that $L = {}^0L = {}^tL$

Remark: To get the B-Matrices, these terms must be transformed to the local η,ξ axes

That‘s it!

Thank you for being patient