Method of Finite Elements II:

Examples 6.14 and 6.15.

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Non-linear analysis

Table 1 Classification of non-linear analysis in solid and structural mechanics (taken from Bathe [1996]).

Type of analysis	Description	Typical formulation used	Stress and strain measures
Materially-nonlinear-only	Infinitesimal displacements and strains; the stress-strain relation is nonlinear	Materially-nonlinear-only (MNO)	Engineering stress and strain
Large displacements, large rotations, but small strains	Displacements and rotations of fibers are large, but fiber extensions and angle changes between fibers are small; the stress-strain relation may be linear or nonlinear	Total Lagrangian (TL)	Second Piola-Kirchhoff stress, Green-Lagrange strain
		Updated Lagrangian (UL)	Cauchy stress, Almansi strain
Large displacements, large rotations, and large strains	Fiber extensions and angle changes between fibers are large, fiber displacements and rotations may also be large; the stress-strain relation may be linear or nonlinear	Total Lagrangian (TL)	Second Piola-Kirchhoff stress, Green-Lagrange strain
		Updated Lagrangian (UL)	Cauchy stress, logarithmic strain

Kinematic description

1. **Total Lagrangian (TL) formulation:**

$$\int_{{}^0V} {}^{t+\Delta t}_{\ \ 0}S_{ij}\,\delta\, {}^{t+\Delta t}_{\ \ 0}e_{ij}\, d\,{}^0V = {}^{t+\Delta t}R\,. \tag{1}$$

2. **Updated Lagrangian (UL) formulation:**

$$\int_{{}^tV} {}^{t+\Delta t}_{\ \ t}S_{ij}\,\delta\, {}^{t+\Delta t}_{\ \ t}e_{ij}\, d\,{}^tV = {}^{t+\Delta t}R\,. \tag{2}$$

Kinematic variables

1. **Stress measures:**
   - Cauchy stress tensor ${}^{t}\boldsymbol{\tau}$
   - Second Piola-Kirchhoff stress tensor:

$$ {}^{t}_{0}\mathbf{S} = \frac{{}^{0}\rho}{{}^{t}\rho}\, {}^{0}_{t}\mathbf{X}\, {}^{t}\boldsymbol{\tau}\, {}^{0}_{t}\mathbf{X}^{\mathsf{T}} \,. \tag{3} $$

2. **Strain measures:**
   - Almansi strain tensor:

$$ {}^{t}_{0}\mathbf{A} = \frac{1}{2}\left(\mathbf{1} - \left({}^{t}_{0}\mathbf{X}\, {}^{t}_{0}\mathbf{X}^{\mathsf{T}}\right)^{-1}\right) \,. \tag{4} $$

   - Green-Lagrange strain tensor:

$$ {}^{t}_{0}\mathbf{E} = \frac{1}{2}\left({}^{t}_{0}\mathbf{X}^{\mathsf{T}}\, {}^{t}_{0}\mathbf{X} - \mathbf{1}\right) \,. \tag{5} $$

Example 6.14

Show: Components of second Piola-Kirchhoff stress tensor are invariant under rigid body rotation of material.

Example 6.15

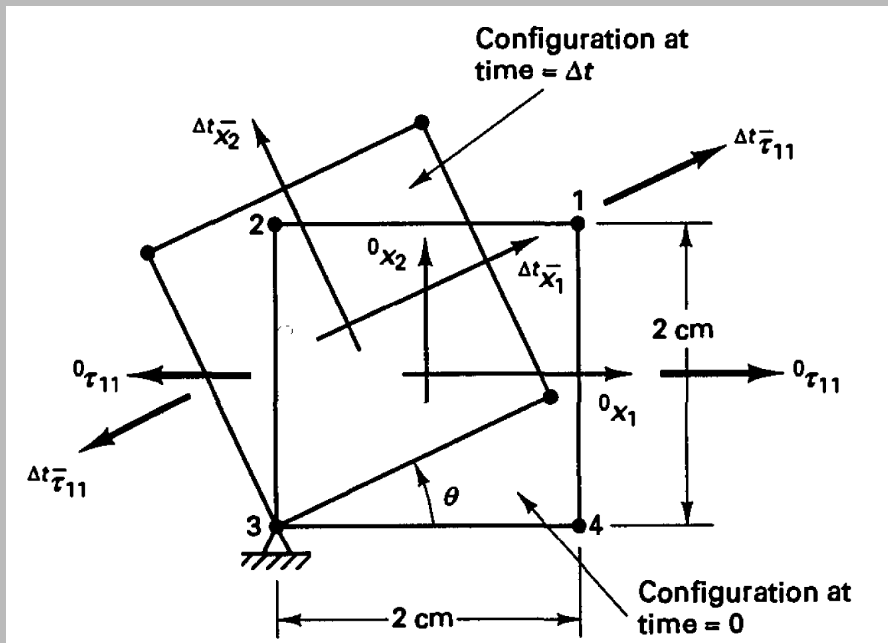

Fig. 1 Four-node element with initial stress subjected to rotation (from Bathe [1996]).

Given: Four-node element with initial stress τ_{11} at time $t = 0$ is subjected to rigid body rotation from time $t = 0$ to time $t = \Delta t$ through angle θ.

Show: Components of second Piola-Kirchhoff stress tensor did not change as a result of rigid body rotation.