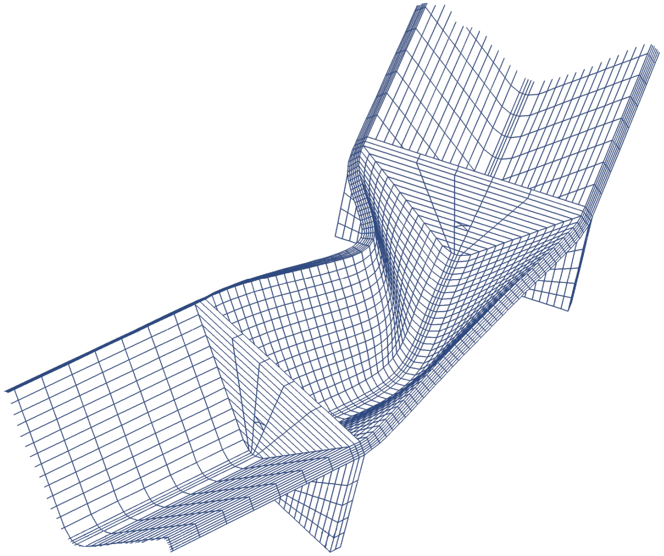
The Finite Element Method for the Analysis of Non-Linear and Dynamic Systems



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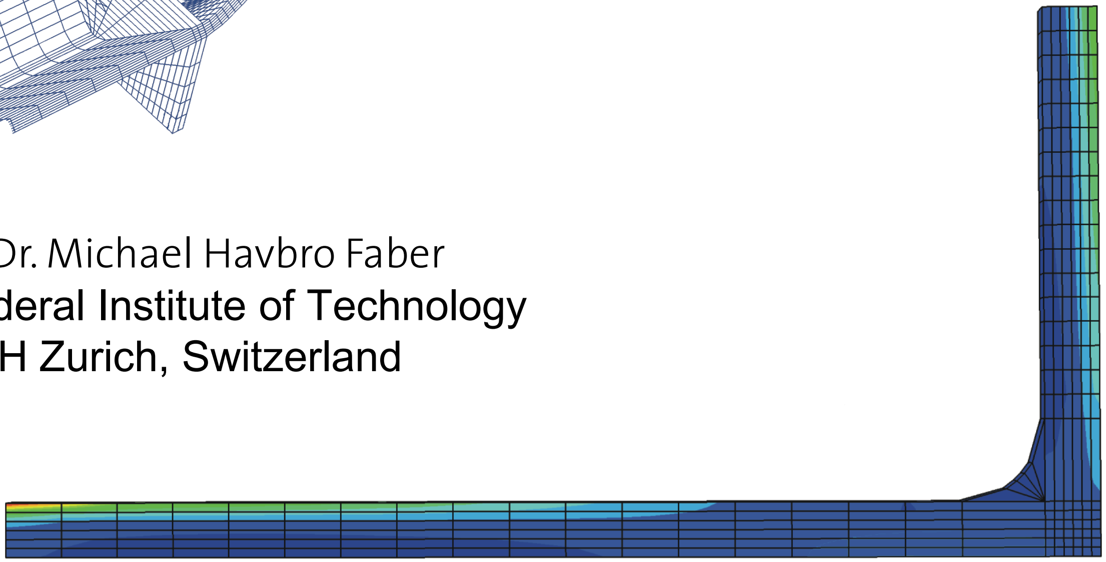
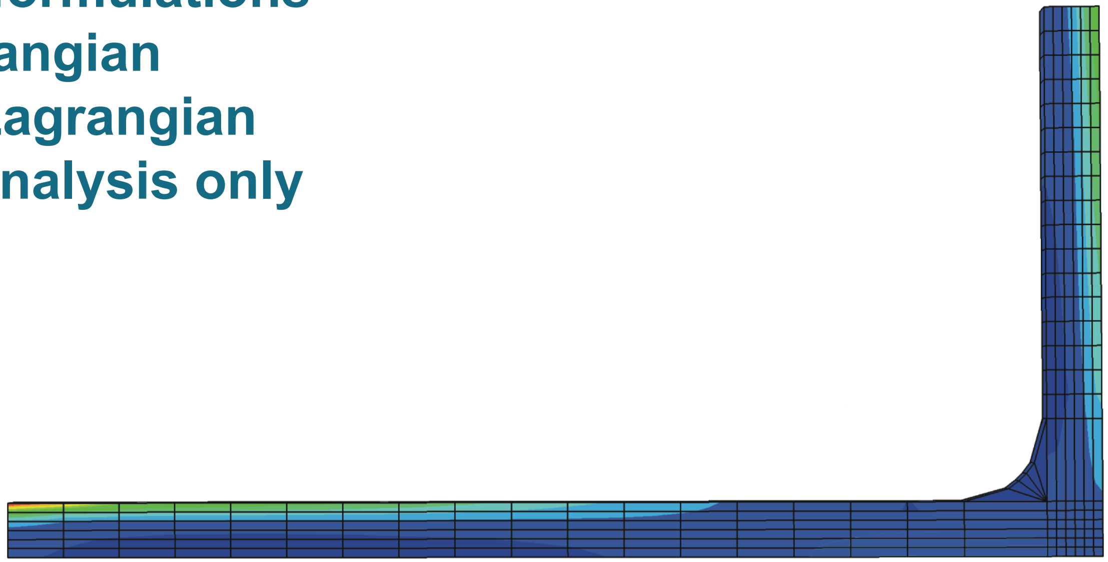


Contents of Today's Lecture

- Short summary of the main findings from the last lecture
- Aim of the present lecture – in short ☺
- The deformation gradient, strain and stress tensors
- Continuum mechanics formulations
  - incremental total Lagrangian
  - incremental updated Lagrangian
  - materially non-linear analysis only



Short summary of the last lecture

- **The basic approach in incremental anaylsis is**

$$ {}^{t+\Delta t}\mathbf{R} - {}^{t+\Delta t}\mathbf{F} = 0 $$

assuming that ${}^{t+\Delta t}\mathbf{R}$ is independent of the deformations we have

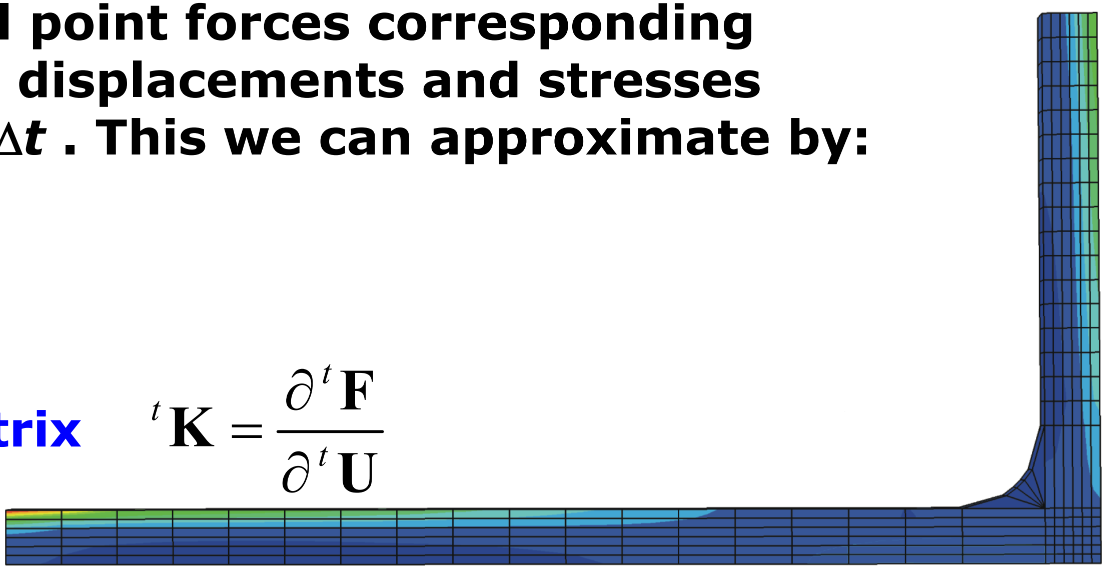
$$ {}^{t+\Delta t}\mathbf{F} = {}^{t}\mathbf{F} + \mathbf{F} $$

We know the solution ^{t}F at time *t* and F is the increment in the nodal point forces corresponding to an increment in the displacements and stresses from time *t* to time *t*+Δ*t* . This we can approximate by:

$$ \mathbf{F} = {}^{t}\mathbf{K}\mathbf{U} $$

Tangent stiffness matrix

$$ {}^{t}\mathbf{K} = \frac{\partial\, {}^{t}\mathbf{F}}{\partial\, {}^{t}\mathbf{U}} $$



Short summary of the last lecture

- **The basic approach in incremental anaylsis is**

We may now substitute the tangent stiffness matrix into the equlibrium relation

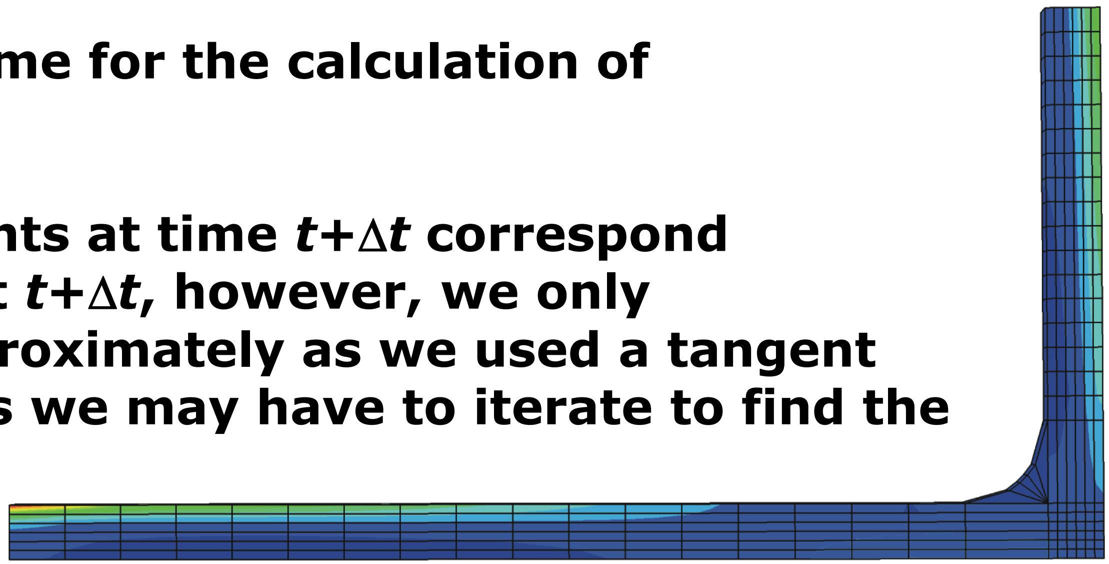
$$ {}^{t}\mathbf{K}\mathbf{U} = {}^{t+\Delta t}\mathbf{R} - {}^{t}\mathbf{F} $$

$$ \Downarrow $$

$$ {}^{t+\Delta t}\mathbf{U} = {}^{t}\mathbf{U} + \mathbf{U} $$

which gives us a scheme for the calculation of the displacements

The exact displacements at time *t*+Δ*t* correspond to the applied loads at *t*+Δ*t*, however, we only determined these approximately as we used a tangent stiffness matrix – thus we may have to iterate to find the solution



Short summary of the last lecture

- **The basic approach in incremental anaylsis is**

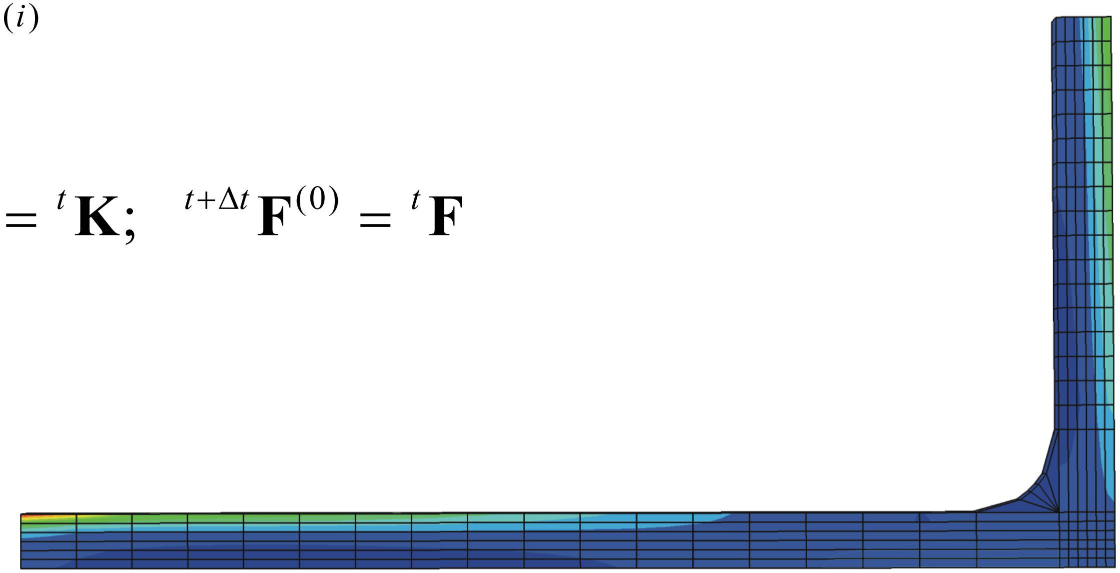
We may use the Newton-Raphson iteration scheme to find the equlibrium within each load increment:

$$ {}^{t+\Delta t}\mathbf{K}^{(i-1)}\Delta\mathbf{U}^{(i)} = {}^{t+\Delta t}\mathbf{R} - {}^{t+\Delta t}\mathbf{F}^{(i-1)} \qquad \textbf{(out of balance load vector)} $$

$$ {}^{t+\Delta t}\mathbf{U}^{(i)} = {}^{t+\Delta t}\mathbf{U}^{(i-1)} + \Delta\mathbf{U}^{(i)} $$

with initial conditions

$$ {}^{t+\Delta t}\mathbf{U}^{(0)} = {}^{t}\mathbf{U}; \quad {}^{t+\Delta t}\mathbf{K}^{(0)} = {}^{t}\mathbf{K}; \quad {}^{t+\Delta t}\mathbf{F}^{(0)} = {}^{t}\mathbf{F} $$



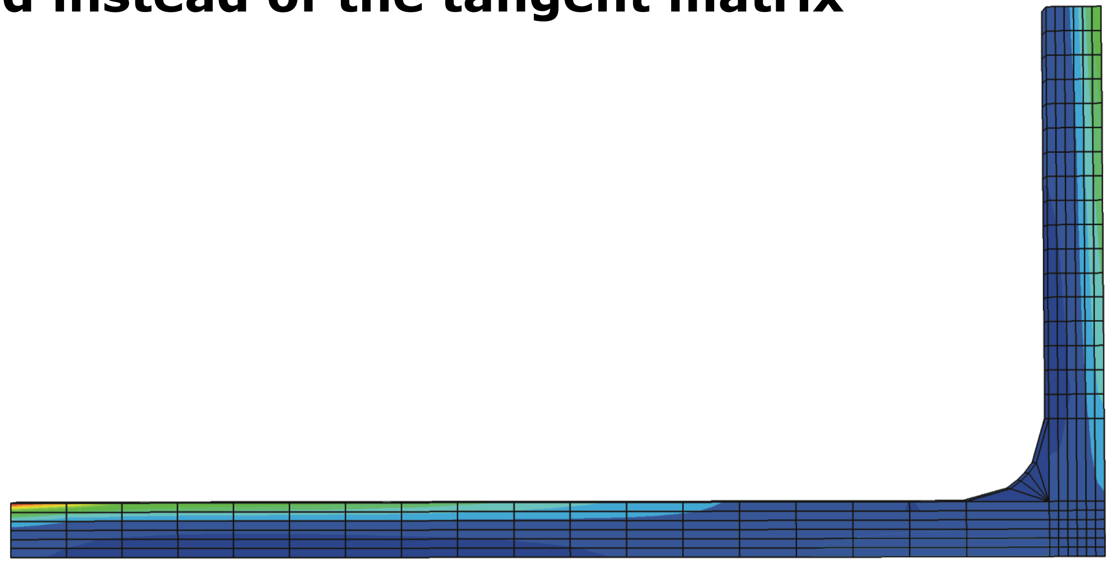
Short summary of the last lecture

- **The basic approach in incremental anaylsis is**

It may be expensive to calculate the tangent stiffness matrix and;

In the Modified Newton-Raphson iteration scheme it is thus only calculated in the beginning of each new load step

In the Quasi-Newton iteration schemes the secant stiffness matrix is used instead of the tangent matrix



Short summary of the last lecture

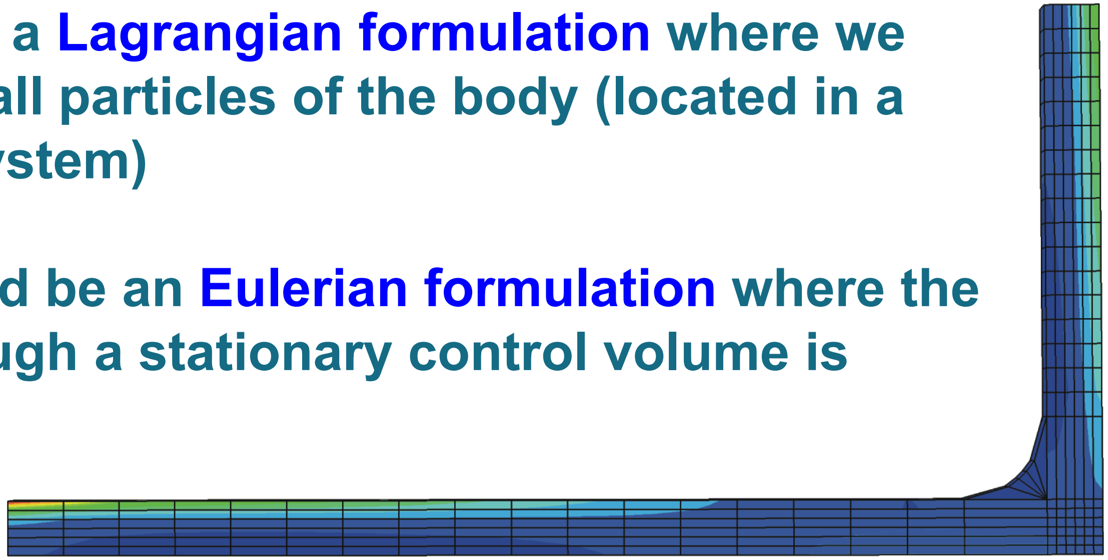
- **The basic problem:**

  We want to establish the solution to a non-linear mechanical problem using an incremental formulation

  The equilibrium must be established for the considered body in its current configuration

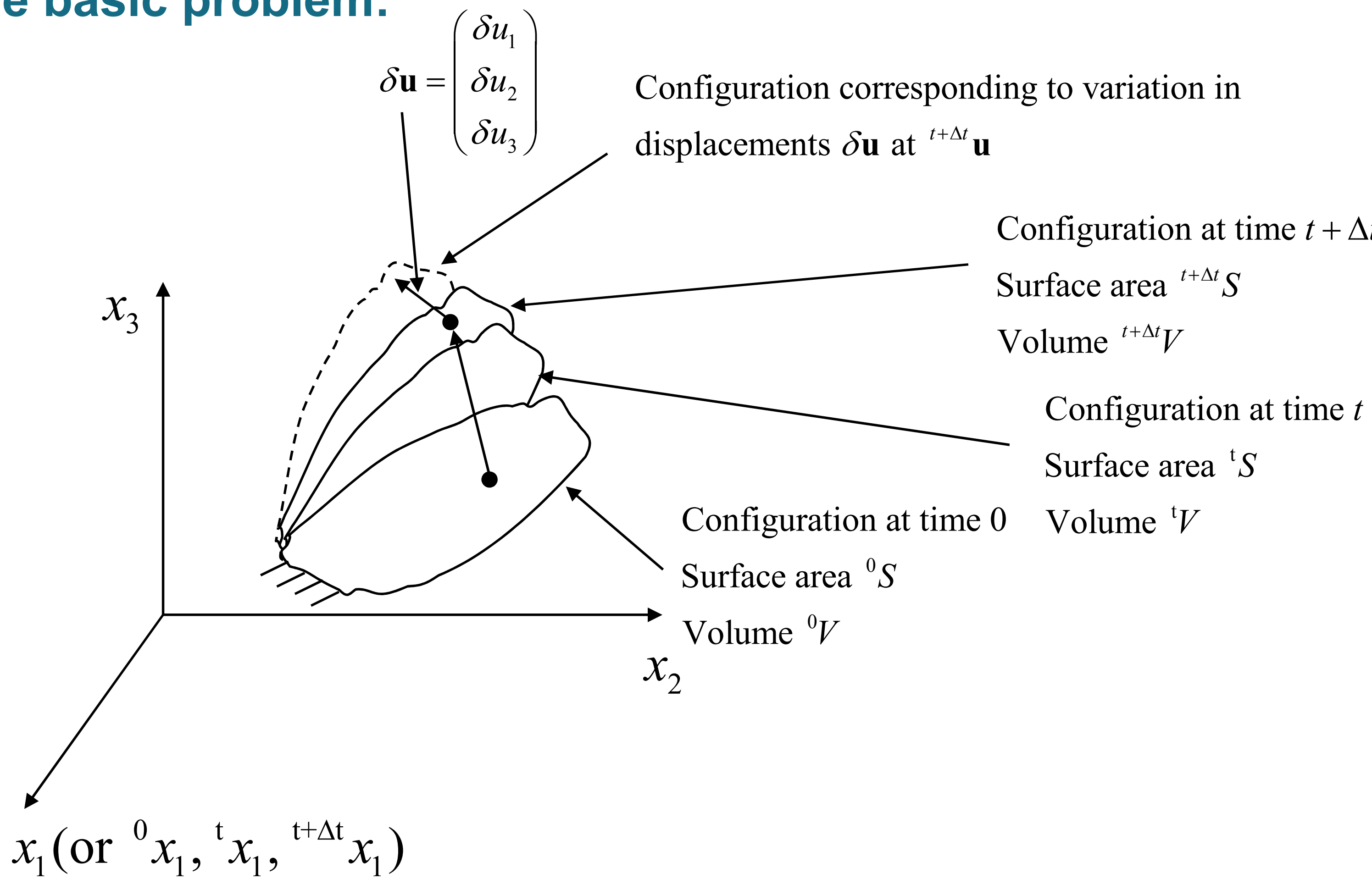
  In proceeding we adopt a Lagrangian formulation where we track the movement of all particles of the body (located in a Cartesian coordinate system)

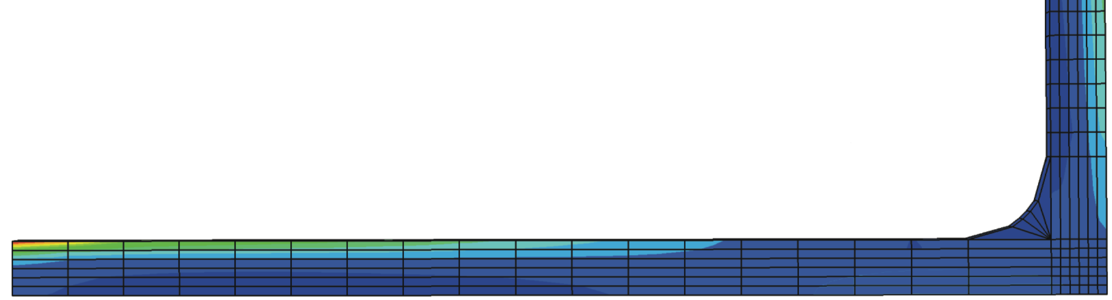
  Another approach would be an Eulerian formulation where the motion of material through a stationary control volume is considered



Short summary of the last lecture

- **The basic problem:**

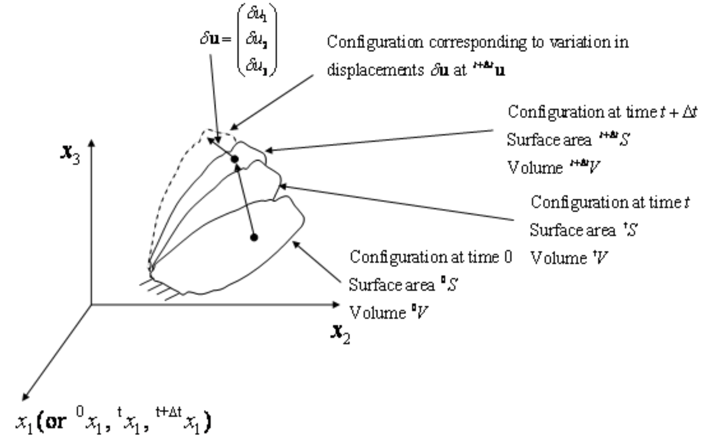




Short summary of the last lecture

- **The Lagrangian formulation**

  We express equilibrium of the body at time *t+Δt* using the principle of virtual displacements



$$\int_{{}^{t+\Delta t}V} {}^{t+\Delta t}\tau \, \delta_{t+\Delta t}e_{ij} \, d\,{}^{t+\Delta t}V = {}^{t+\Delta t}R$$

${}^{t+\Delta t}\tau$: Cartesian components of the Cauchy stress tensor

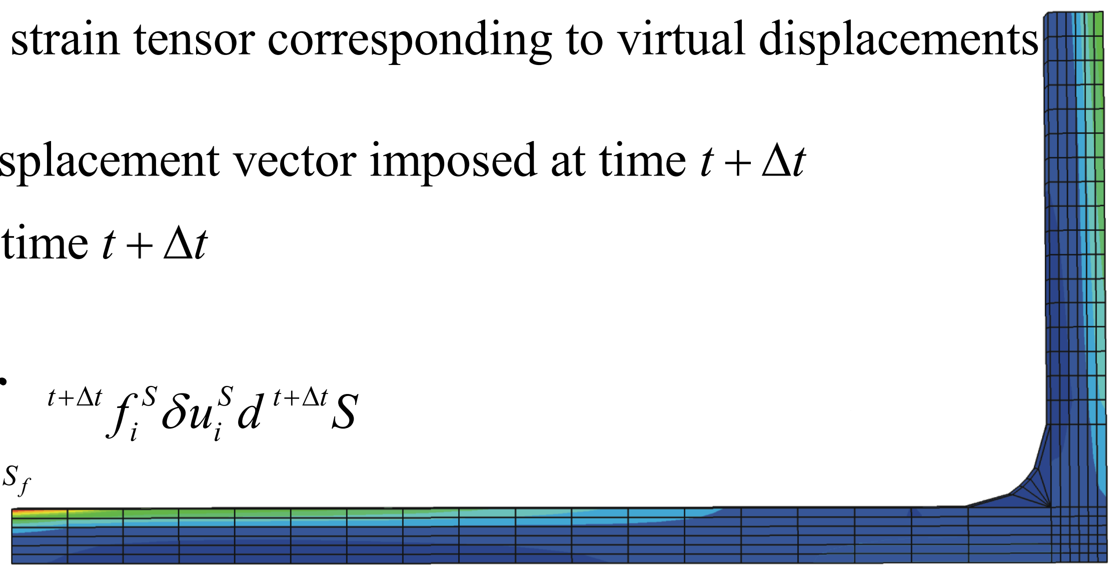
$$\delta_{t+\Delta t}e_{ij} = \frac{1}{2}\left(\frac{\partial \delta u_i}{\partial\,{}^{t+\Delta t}x_j} + \frac{\partial \delta u_j}{\partial\,{}^{t+\Delta t}x_i}\right) = \text{strain tensor corresponding to virtual displacements}$$

δu_i : Components of virtual displacement vector imposed at time $t+\Delta t$

${}^{t+\Delta t}x_i$: Cartesian coordinate at time $t+\Delta t$

${}^{t+\Delta t}V$: Volume at time $t+\Delta t$

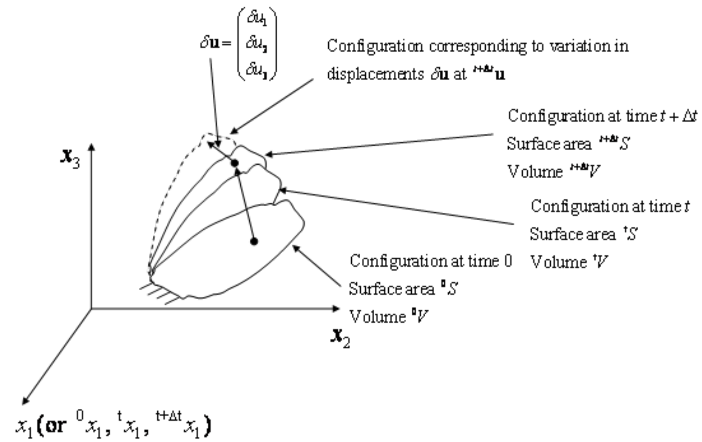
$$ {}^{t+\Delta t}R = \int_{{}^{t+\Delta t}V} {}^{t+\Delta t}f_i^B \, \delta u_i \, d\,{}^{t+\Delta t}V + \int_{{}^{t+\Delta t}S_f} {}^{t+\Delta t}f_i^S \, \delta u_i^S \, d\,{}^{t+\Delta t}S$$



Short summary of the last lecture

- **The Lagrangian formulation**

  We express equilibrium of the body at time *t*+Δ*t* using the principle of virtual displacements



$$ {}^{t+\Delta t}R = \int_{{}^{t+\Delta t}V} {}^{t+\Delta t}f_i^B \, \delta u_i \, d\,{}^{t+\Delta t}V + \int_{{}^{t+\Delta t}S_f} {}^{t+\Delta t}f_i^S \, \delta u_i^S \, d\,{}^{t+\Delta t}S $$

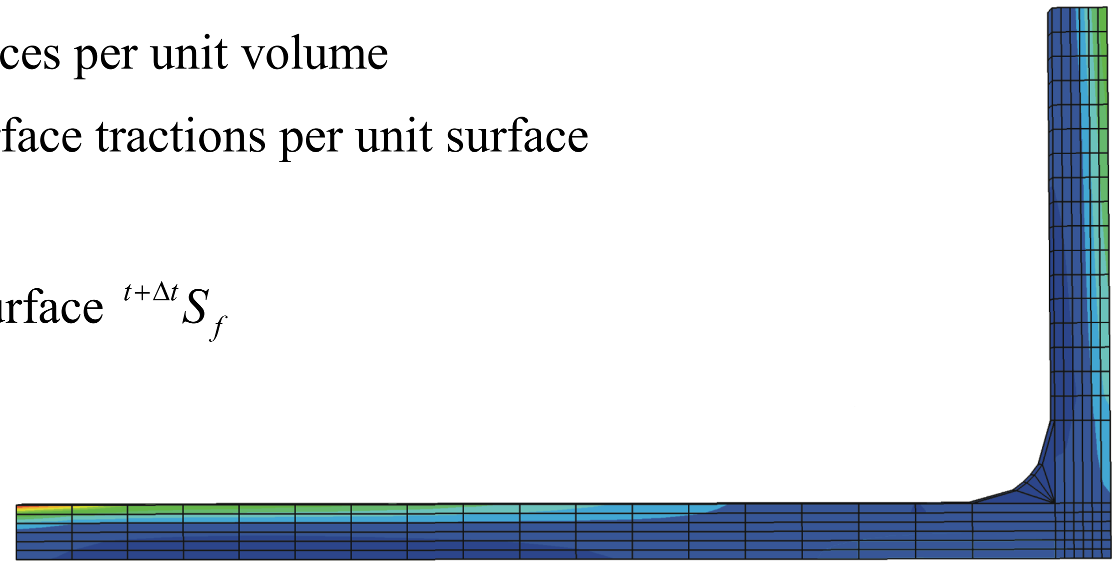
where

${}^{t+\Delta t}f_i^B$: externally apllied forces per unit volume

${}^{t+\Delta t}f_i^S$: externally apllied surface tractions per unit surface

${}^{t+\Delta t}S_f$: surface at time $t+\Delta t$

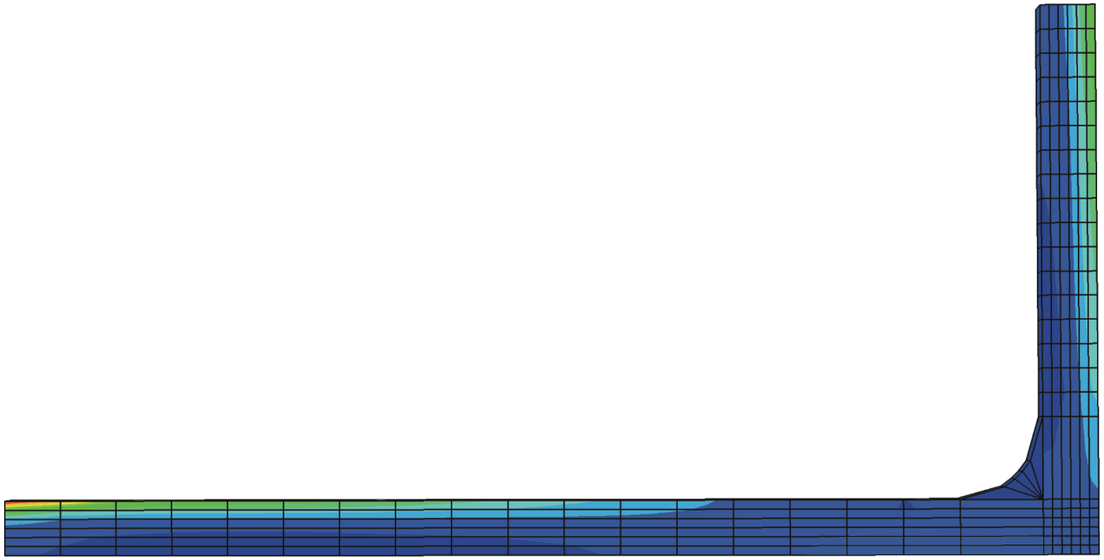
δu_i^S : δu_i evaluated at the surface ${}^{t+\Delta t}S_f$



Short summary of the last lecture

- **The Lagrangian formulation**

  We recognize that our derivations from linear finite element theory are unchanged – but applied to the body in the configuration at time $t+\Delta t$



Short summary of the last lecture

- **In the further we introduce an appropriate notation:**

Coordinates and displacements are related as:

$$^{t}x_i = {}^{0}x_i + {}^{t}u_i$$

$$^{t+\Delta t}x_i = {}^{0}x_i + {}^{t+\Delta t}u_i$$

Increments in displacements are related as:

$$_{t}u_i = {}^{t+\Delta t}u_i - {}^{t}u_i$$

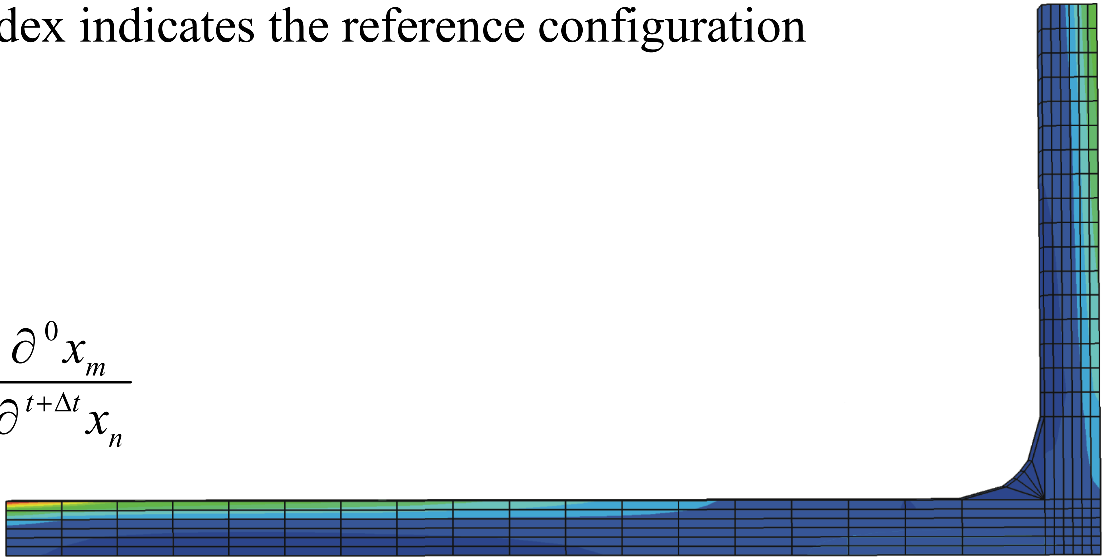
Reference configurations are indexed as e.g.:

$^{t+\Delta t}_{\;\;0}f_i^S$ where the lower left index indicates the reference configuration

$$^{t+\Delta t}\tau_{ij} = {}^{t+\Delta t}_{t+\Delta t}\tau_{ij}$$

Differentiation is indexed as:

$$^{t+\Delta t}_{\;\;0}u_{i,j} = \frac{\partial\, ^{t+\Delta t}u_i}{\partial\, ^{0}x_j}, \qquad {}^{\;\;0}_{t+\Delta t}x_{m,n} = \frac{\partial\, ^{0}x_m}{\partial\, ^{t+\Delta t}x_n}$$



Aim of the present lecture

- **We have already formulated the continuum mechanich incremental equations of motion**

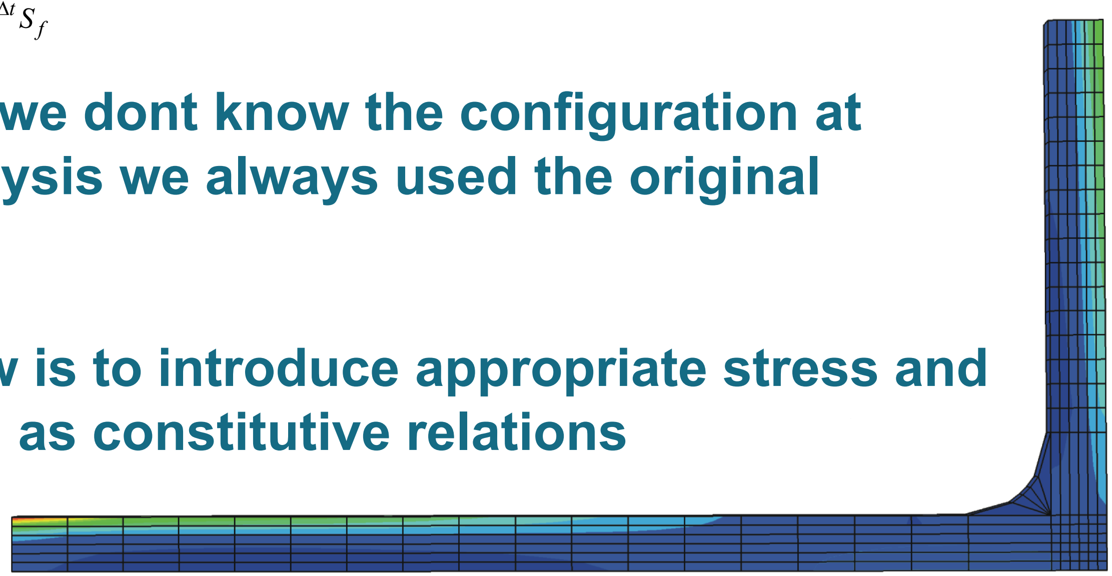
$$\int_{{}^{t+\Delta t}V} {}^{t+\Delta t}\tau \delta_{t+\Delta t} e_{ij} d\,{}^{t+\Delta t}V = {}^{t+\Delta t}R$$

and

$${}^{t+\Delta t}R = \int_{{}^{t+\Delta t}V} {}^{t+\Delta t}f_i^B \delta u_i d\,{}^{t+\Delta t}V + \int_{{}^{t+\Delta t}S_f} {}^{t+\Delta t}f_i^S \delta u_i^S d\,{}^{t+\Delta t}S$$

a basic problem is that we dont know the configuration at time $t+\Delta t$ (in linear analysis we always used the original configuration as basis)

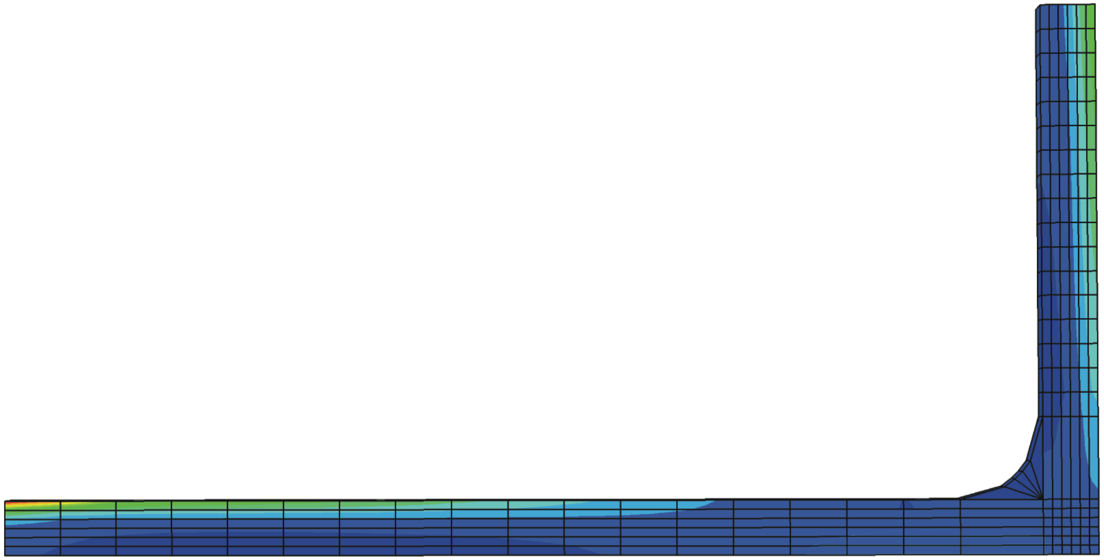
what we need to do now is to introduce appropriate stress and strain measures as well as constitutive relations



The deformation gradient, strain and stress tensors

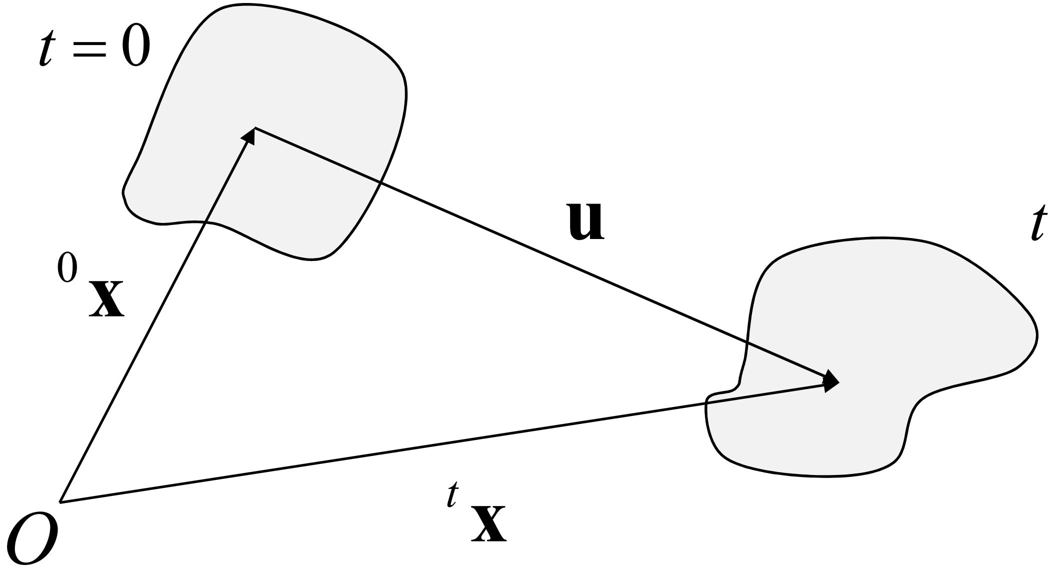
- **As mentioned – we must try to establish a description of the volume we consider such that we can express the internal virtual work in terms of an integral over a volume we know!**

- **Further we would like to be able to decompose the stresses and strains in an efficient manner – keeping track of how the volume stretches and how it rotates (rigidly).**

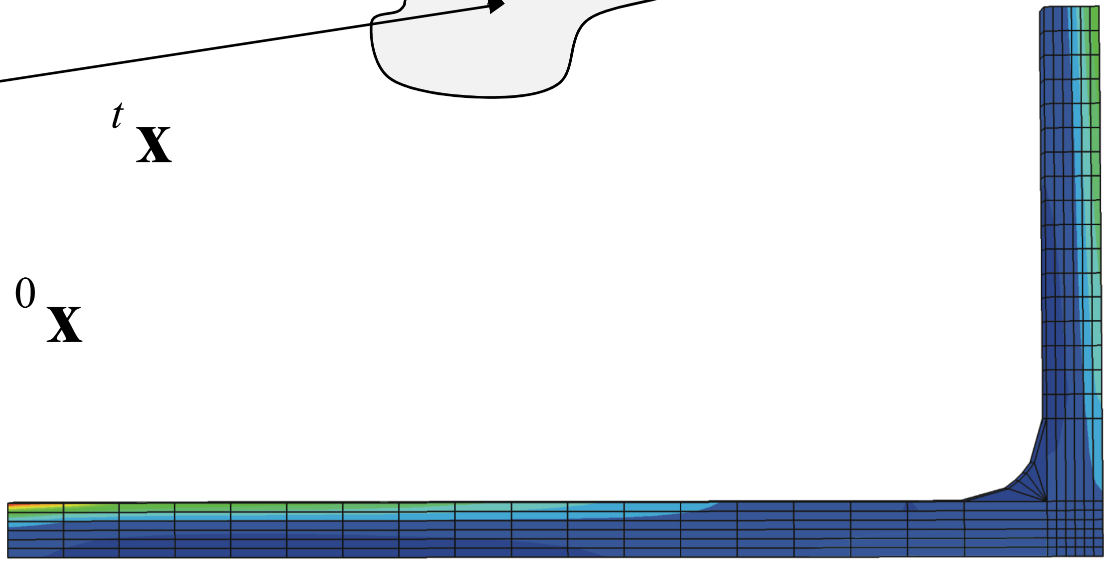


The deformation gradient, strain and stress tensors

We consider a body under deformation at times 0 and *t*

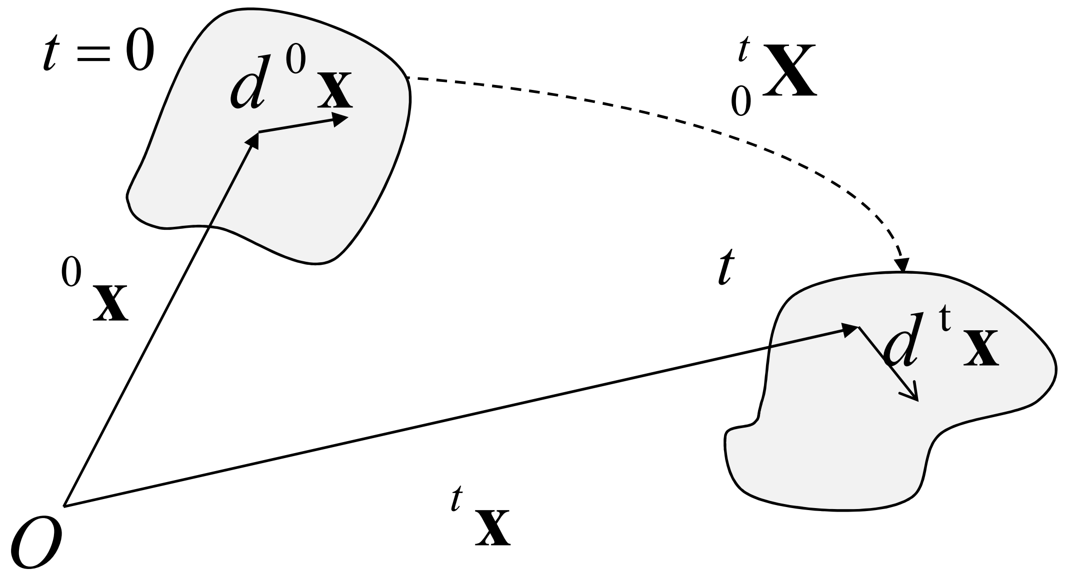


$$\mathbf{u} = {}^{t}\mathbf{x} - {}^{0}\mathbf{x}$$



The deformation gradient, strain and stress tensors

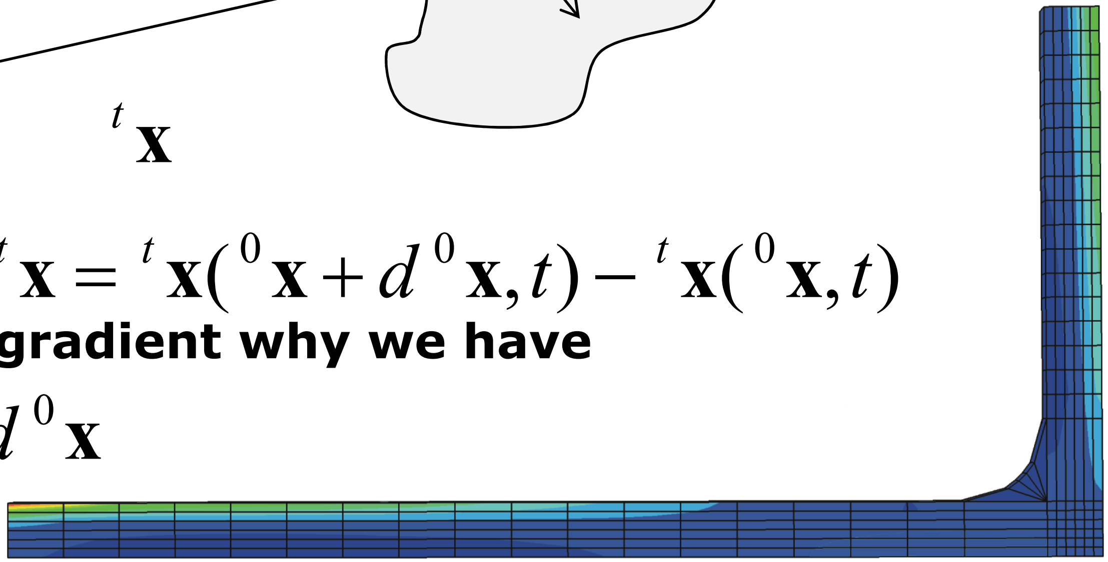
We now consider the change of an infinitesimal gradient vector



The we can write $d\,{}^{t}\mathbf{x} = {}^{t}\mathbf{x}({}^{0}\mathbf{x} + d\,{}^{0}\mathbf{x}, t) - {}^{t}\mathbf{x}({}^{0}\mathbf{x}, t)$

which is linear in the gradient why we have

$$d\,{}^{t}\mathbf{x} = {}^{t}_{0}\mathbf{X} d\,{}^{0}\mathbf{x}$$



The deformation gradient, strain and stress tensors

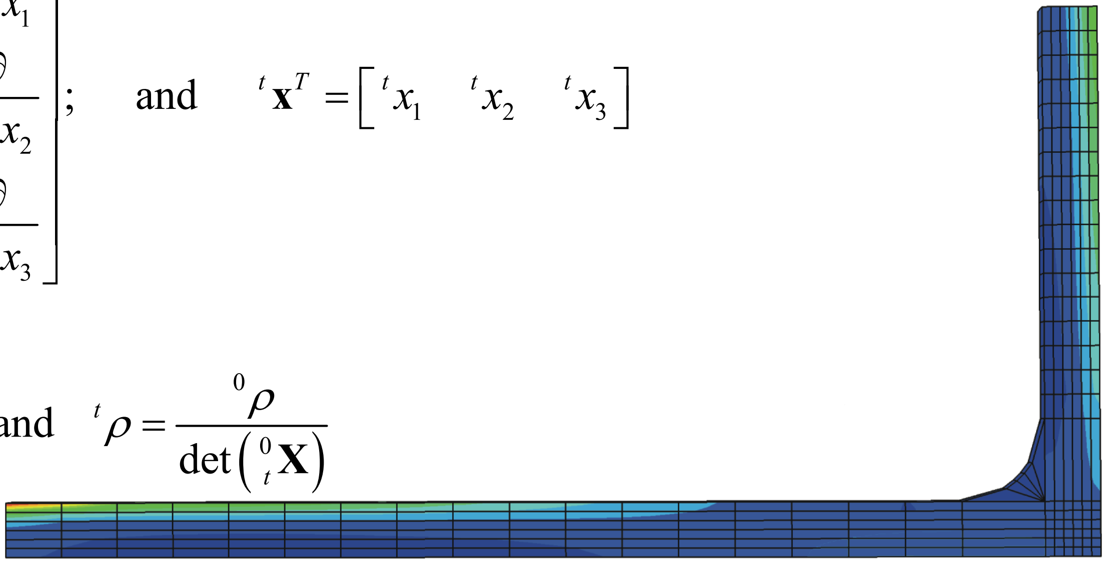
- **We can write the deformation gradient as**

$$ {}_0^t\mathbf{X} = \begin{bmatrix} \dfrac{\partial\, {}^t x_1}{\partial\, {}^0 x_1} & \dfrac{\partial\, {}^t x_1}{\partial\, {}^0 x_2} & \dfrac{\partial\, {}^t x_1}{\partial\, {}^0 x_3} \\[2ex] \dfrac{\partial\, {}^t x_2}{\partial\, {}^0 x_1} & \dfrac{\partial\, {}^t x_2}{\partial\, {}^0 x_2} & \dfrac{\partial\, {}^t x_2}{\partial\, {}^0 x_3} \\[2ex] \dfrac{\partial\, {}^t x_3}{\partial\, {}^0 x_1} & \dfrac{\partial\, {}^t x_3}{\partial\, {}^0 x_2} & \dfrac{\partial\, {}^t x_3}{\partial\, {}^0 x_3} \end{bmatrix} $$

The deformation gradient describes the stretches and rotations that the material fibers have undergone from time zero to time *t*

$$ {}_0^t\mathbf{X} = \left({}_0\nabla\, {}^t\mathbf{x}^T\right)^T, \text{ where } \quad {}_0\nabla = \begin{bmatrix} \dfrac{\partial}{\partial\, {}^0 x_1} \\[2ex] \dfrac{\partial}{\partial\, {}^0 x_2} \\[2ex] \dfrac{\partial}{\partial\, {}^0 x_3} \end{bmatrix}; \quad \text{and} \quad {}^t\mathbf{x}^T = \left[\, {}^t x_1 \quad {}^t x_2 \quad {}^t x_3 \,\right] $$

it can be show that $ {}_0^t\mathbf{X} = \left({}_t^0\mathbf{X}\right)^{-1} $ and $ {}^t\rho = \dfrac{{}^0\rho}{\det\left({}_t^0\mathbf{X}\right)} $



The deformation gradient, strain and stress tensors

- **Then we introduce the Cauchy-Green deformation tensor**

$$
{}_0^t\mathbf{X} = \begin{bmatrix} \frac{\partial\, {}^t x_1}{\partial\, {}^0 x_1} & \frac{\partial\, {}^t x_1}{\partial\, {}^0 x_2} & \frac{\partial\, {}^t x_1}{\partial\, {}^0 x_3} \\ \frac{\partial\, {}^t x_2}{\partial\, {}^0 x_1} & \frac{\partial\, {}^t x_2}{\partial\, {}^0 x_2} & \frac{\partial\, {}^t x_2}{\partial\, {}^0 x_3} \\ \frac{\partial\, {}^t x_3}{\partial\, {}^0 x_1} & \frac{\partial\, {}^t x_3}{\partial\, {}^0 x_2} & \frac{\partial\, {}^t x_3}{\partial\, {}^0 x_3} \end{bmatrix}
$$

$$
{}_0^t\mathbf{X} = ({}_0\nabla\, {}^t\mathbf{x}^T)^T, \text{ where } \quad {}_0\nabla = \begin{bmatrix} \frac{\partial}{\partial\, {}^0 x_1} \\ \frac{\partial}{\partial\, {}^0 x_2} \\ \frac{\partial}{\partial\, {}^0 x_3} \end{bmatrix}; \quad \text{and} \quad {}^t\mathbf{x}^T = \begin{bmatrix} {}^t x_1 & {}^t x_2 & {}^t x_3 \end{bmatrix}
$$

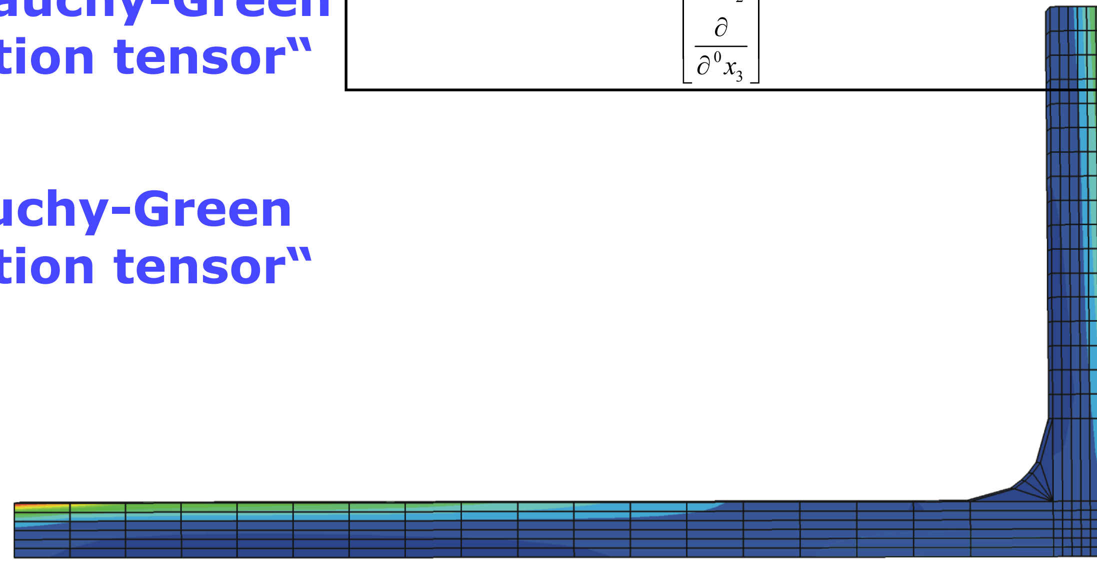
The deformation gradient is also used to measure the stretch of a material fiber and the change in angle between fibers due to the deformation

$$
{}_0^t\mathbf{C} = {}_0^t\mathbf{X}^T \, {}_0^t\mathbf{X}
$$

"Right Cauchy-Green deformation tensor"

$$
{}_0^t\mathbf{B} = {}_0^t\mathbf{X} \, {}_0^t\mathbf{X}^T
$$

"Left Cauchy-Green deformation tensor"



The deformation gradient, strain and stress tensors

- **The deformation gradient**

The deformation gradient can be decomposed into a unique product of two matrices

$$ {}_0^t\mathbf{X} = \begin{bmatrix} \frac{\partial\, {}^t x_1}{\partial\, {}^0 x_1} & \frac{\partial\, {}^t x_1}{\partial\, {}^0 x_2} & \frac{\partial\, {}^t x_1}{\partial\, {}^0 x_3} \\ \frac{\partial\, {}^t x_2}{\partial\, {}^0 x_1} & \frac{\partial\, {}^t x_2}{\partial\, {}^0 x_2} & \frac{\partial\, {}^t x_2}{\partial\, {}^0 x_3} \\ \frac{\partial\, {}^t x_3}{\partial\, {}^0 x_1} & \frac{\partial\, {}^t x_3}{\partial\, {}^0 x_2} & \frac{\partial\, {}^t x_3}{\partial\, {}^0 x_3} \end{bmatrix} $$

$$ {}_0^t\mathbf{X} = ({}_0\nabla\, {}^t\mathbf{x}^T)^T, \text{ where } \quad {}_0\nabla = \begin{bmatrix} \frac{\partial}{\partial\, {}^0 x_1} \\ \frac{\partial}{\partial\, {}^0 x_2} \\ \frac{\partial}{\partial\, {}^0 x_3} \end{bmatrix}; \quad \text{and} \quad {}^t\mathbf{x}^T = \begin{bmatrix} {}^t x_1 & {}^t x_2 & {}^t x_3 \end{bmatrix} $$

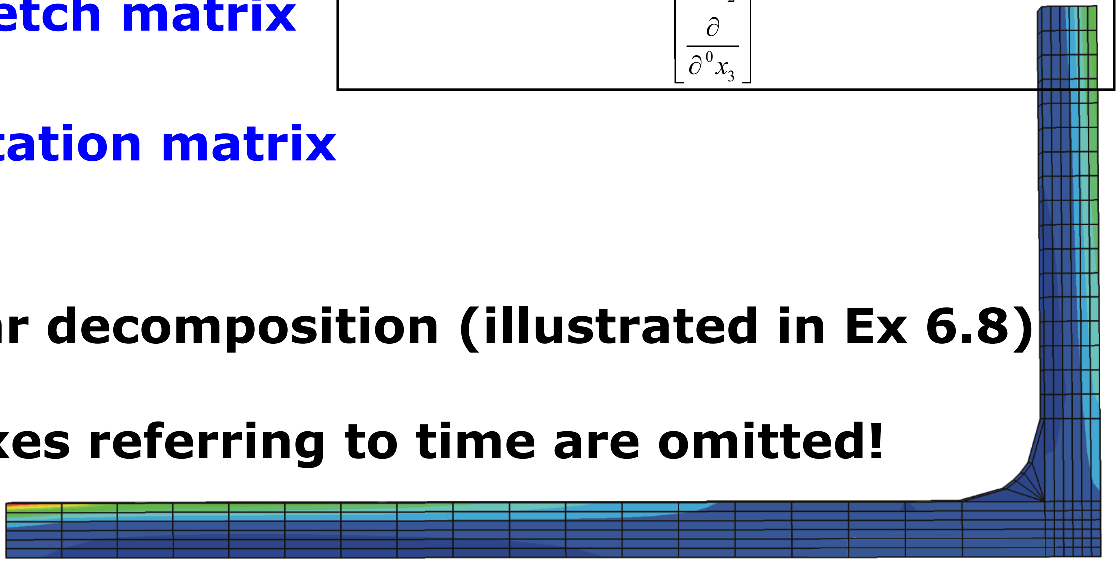
$$ {}_0^t\mathbf{X} = {}_0^t\mathbf{R}\, {}_0^t\mathbf{U} $$

${}_0^t\mathbf{U}$ **Symmetric stretch matrix**

${}_0^t\mathbf{R}$ **Orthogonal rotation matrix**

Referred to as a polar decomposition (illustrated in Ex 6.8)

Sometimes the indexes referring to time are omitted!



The deformation gradient, strain and stress tensors

- **Decomposition of the deformation gradient**

We continue by rewriting the deformation gradient

$$\mathbf{X} = \mathbf{R}\mathbf{U} = \mathbf{R}\mathbf{U}\mathbf{R}^T\mathbf{R} = \mathbf{V}\mathbf{R}$$

${}_0^t\mathbf{X} = {}_0^t\mathbf{R}\,{}_0^t\mathbf{U}$	
${}_0^t\mathbf{U}$	Symmetric stretch matrix
${}_0^t\mathbf{R}$	Orthogonal rotation matrix

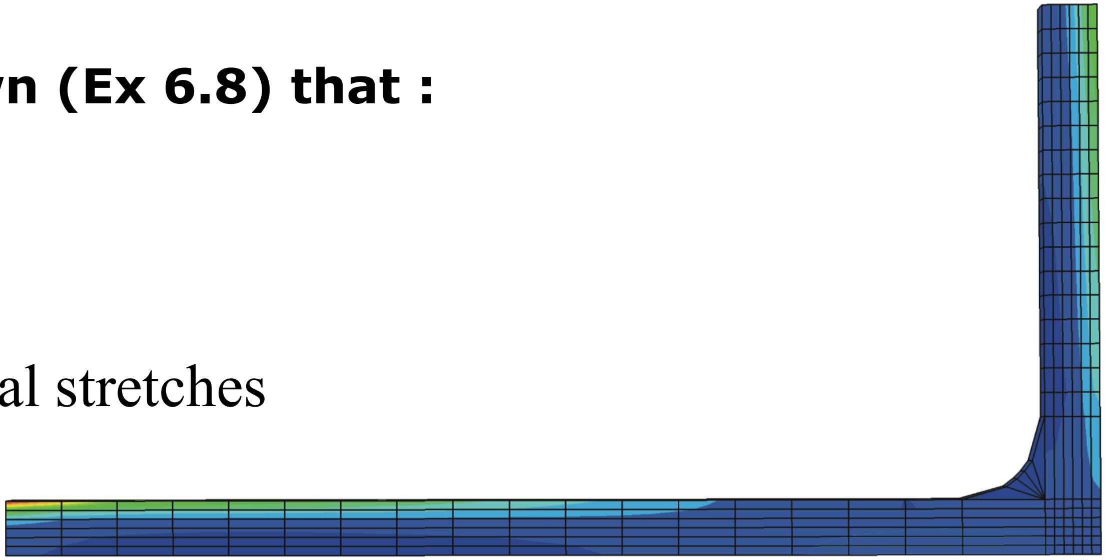
U: right stretch matrix
V: left stretch matrix

Further it can be shown (Ex 6.8) that :

$$\mathbf{U} = \mathbf{R}_L \mathbf{\Lambda} \mathbf{R}_L^T$$

$\mathbf{\Lambda}$: Principal stretches

$\mathbf{R}_L$: Direction of principal stretches



The deformation gradient, strain and stress tensors

- **Decomposition of the deformation gradient**

There is also:

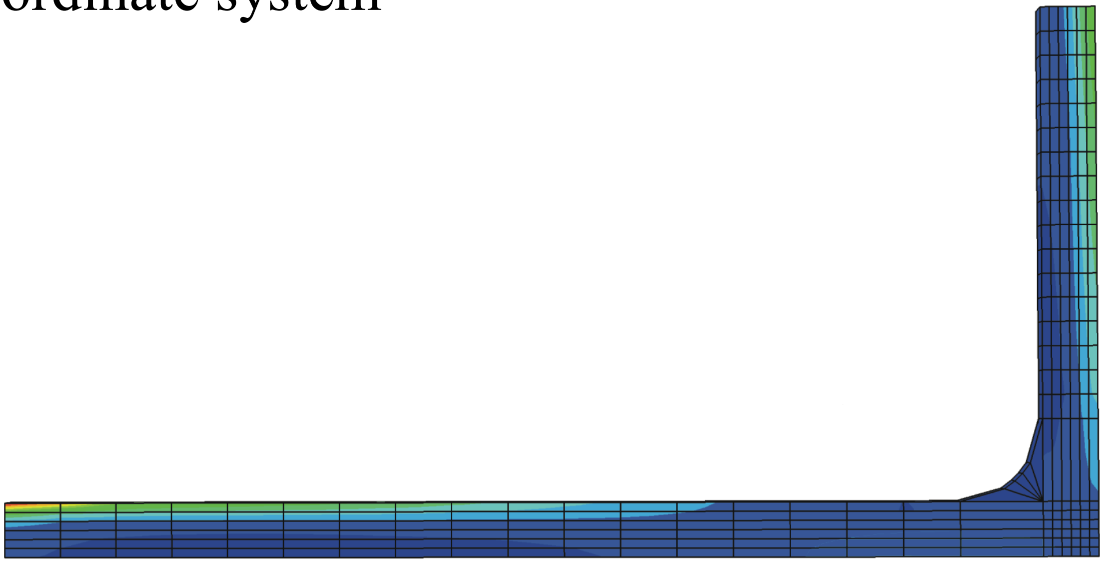
$$\mathbf{V} = \mathbf{R}_E \mathbf{\Lambda} \mathbf{R}_E^T$$

$\mathbf{R}_E$: Base vectors of principal stretches in the stationary coordinate system

$$\mathbf{U} = \mathbf{R}_L \mathbf{\Lambda} \mathbf{R}_L^T$$

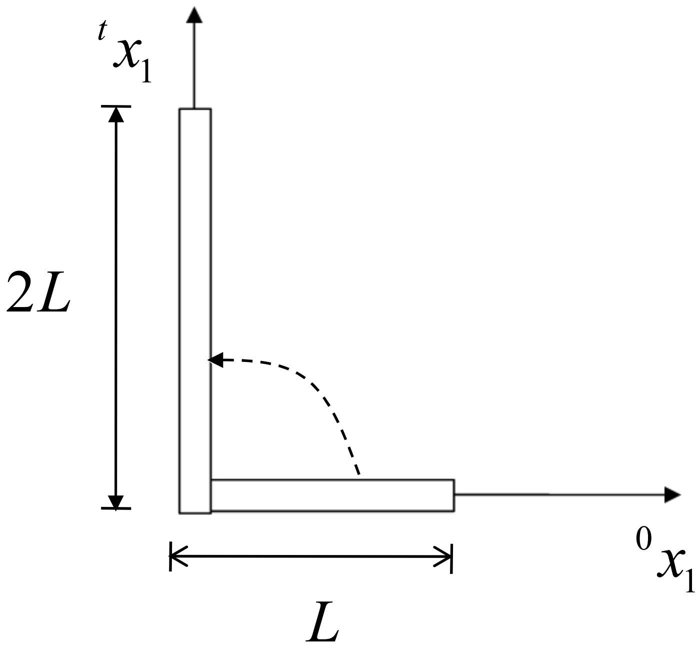
$\mathbf{\Lambda}$: Principal stretches

$\mathbf{R}_L$: Direction of principal stretches



The deformation gradient, strain and stress tensors

We consider a bar under stretch and rotation



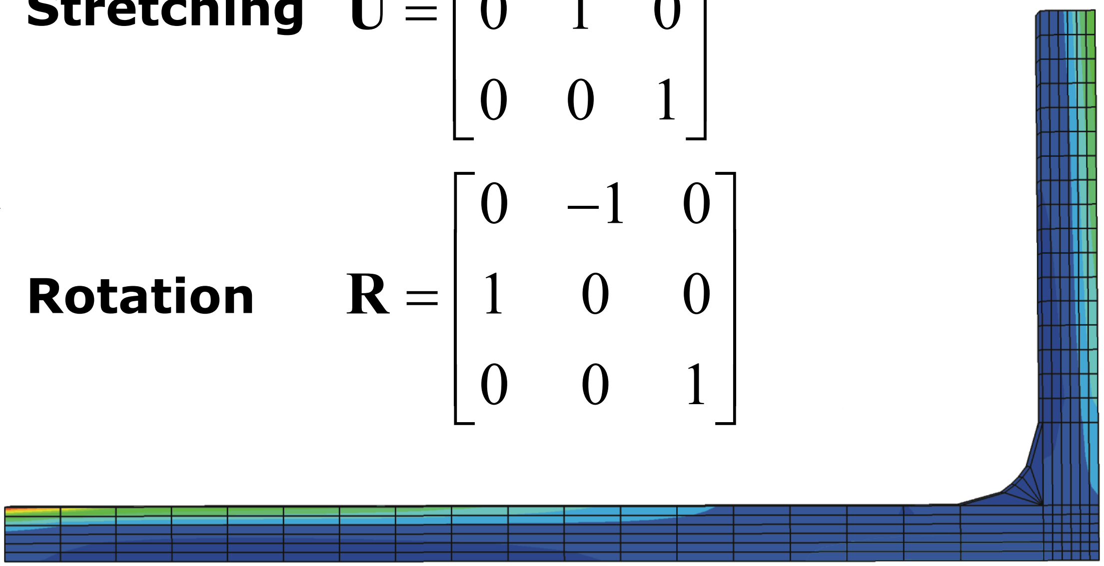
$$\mathbf{X} = \mathbf{R}\mathbf{U}$$

Decomposition (Ex 6.8)

It is instructive to consider the deformation in two steps

Stretching $\mathbf{U} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Rotation $\mathbf{R} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$



The deformation gradient, strain and stress tensors

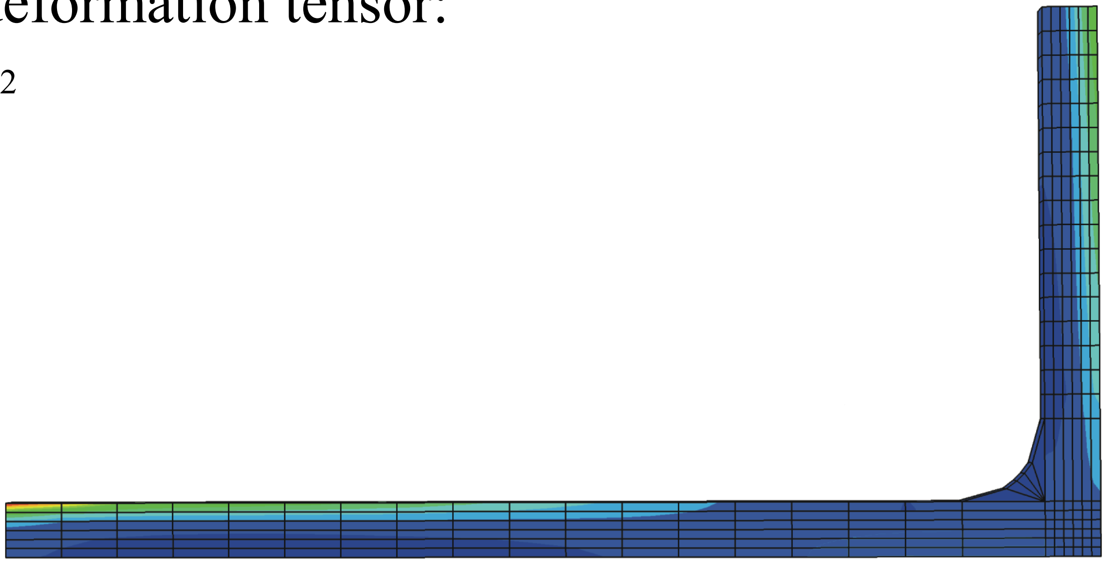
- **Using the decomposition of the deformation gradient we may rewrite the right and left Cauchy-Green deformation tensors:**

The right Cauchy-Green deformation tensor:

$$\mathbf{C} = \mathbf{X}^T \mathbf{X} = \mathbf{U}\mathbf{R}^T \mathbf{R}\mathbf{U} = \mathbf{U}^2$$

The left Cauchy-Green deformation tensor:

$$\mathbf{B} = \mathbf{X}\mathbf{X}^T = \mathbf{V}\mathbf{R}\mathbf{R}^T \mathbf{V} = \mathbf{V}^2$$



The deformation gradient, strain and stress tensors

- **We now proceed from deformations to strains ☺**

The strain may be understood as the stretch per unit length why we can assess the strain through the inner product between two infinitesimal vectors before and after deformation

$$d^{t}\mathbf{x}_1 \bullet d^{t}\mathbf{x}_2 - d^{0}\mathbf{x}_1 \bullet d^{0}\mathbf{x}_2 = (\mathbf{X} d^{0}\mathbf{x}_1) \bullet (\mathbf{X} d^{0}\mathbf{x}_2) - d^{0}\mathbf{x}_1 \bullet d^{0}\mathbf{x}_2$$

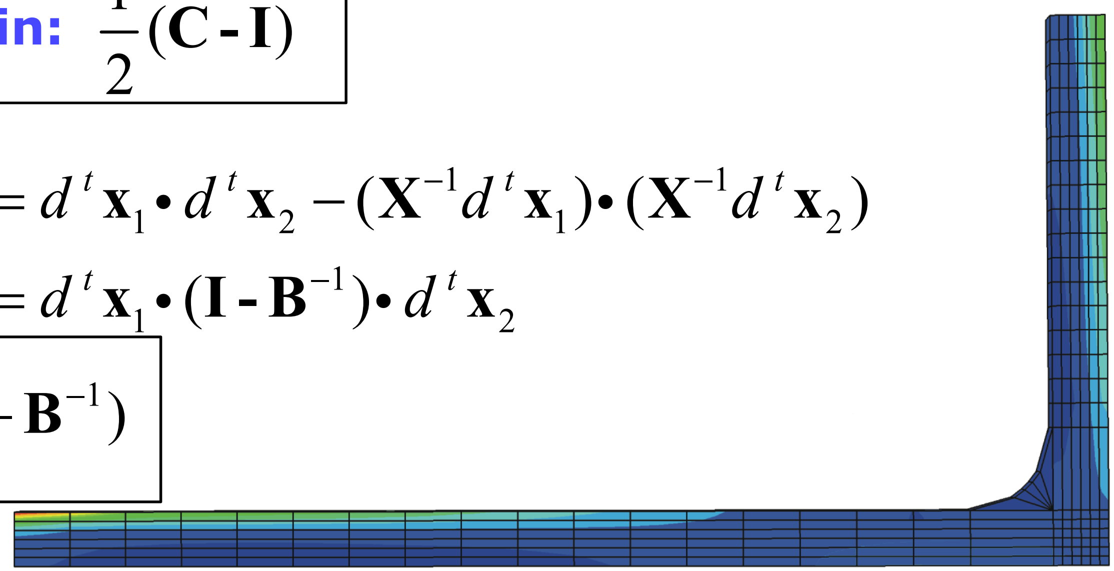
$$= d^{0}\mathbf{x}_1 \bullet (\mathbf{C} - \mathbf{I}) \bullet d^{0}\mathbf{x}_2$$

Green-Lagrange strain: $\frac{1}{2}(\mathbf{C} - \mathbf{I})$

$$d^{t}\mathbf{x}_1 \bullet d^{t}\mathbf{x}_2 - d^{0}\mathbf{x}_1 \bullet d^{0}\mathbf{x}_2 = d^{t}\mathbf{x}_1 \bullet d^{t}\mathbf{x}_2 - (\mathbf{X}^{-1} d^{t}\mathbf{x}_1) \bullet (\mathbf{X}^{-1} d^{t}\mathbf{x}_2)$$

$$= d^{t}\mathbf{x}_1 \bullet (\mathbf{I} - \mathbf{B}^{-1}) \bullet d^{t}\mathbf{x}_2$$

Almansi strain: $\frac{1}{2}(\mathbf{I} - \mathbf{B}^{-1})$



The deformation gradient, strain and stress tensors

- **Lets see an example (one-dimensional)**

We assume the following deformation gradient matrix

$$\mathbf{X} = \begin{bmatrix} \frac{l}{L} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \qquad \text{i.e. pure stretch}$$

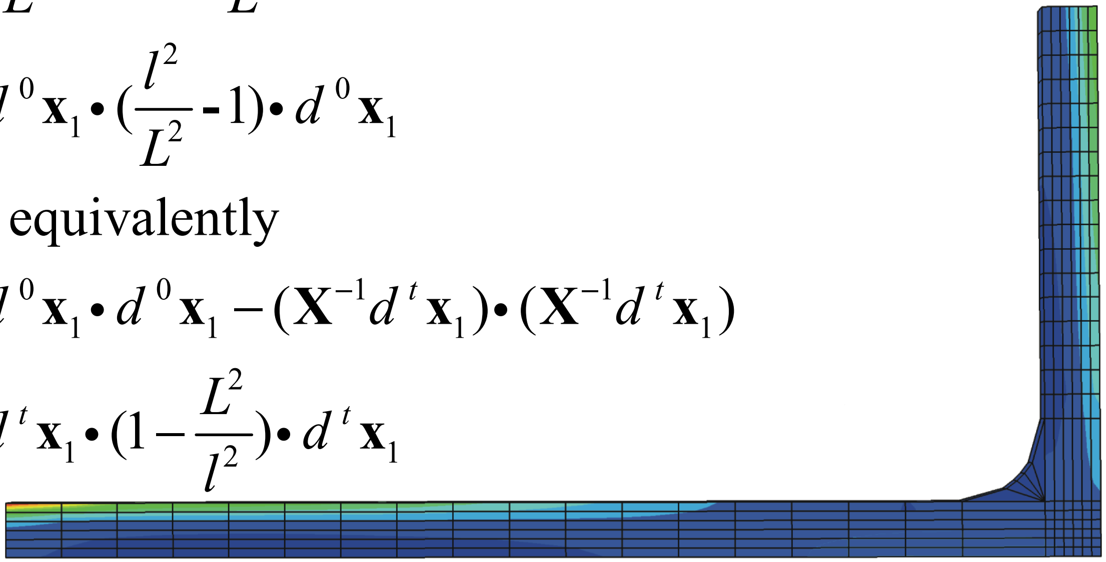
$$d\,^t\mathbf{x}_1 \bullet d\,^t\mathbf{x}_1 - d\,^0\mathbf{x}_1 \bullet d\,^0\mathbf{x}_1 = (\frac{l}{L} d\,^0\mathbf{x}_1) \bullet (\frac{l}{L} d\,^0\mathbf{x}_1) - d\,^0\mathbf{x}_1 \bullet d\,^0\mathbf{x}_1$$

$$= d\,^0\mathbf{x}_1 \bullet (\frac{l^2}{L^2} - 1) \bullet d\,^0\mathbf{x}_1$$

or equivalently

$$= d\,^0\mathbf{x}_1 \bullet d\,^0\mathbf{x}_1 - (\mathbf{X}^{-1} d\,^t\mathbf{x}_1) \bullet (\mathbf{X}^{-1} d\,^t\mathbf{x}_1)$$

$$= d\,^t\mathbf{x}_1 \bullet (1 - \frac{L^2}{l^2}) \bullet d\,^t\mathbf{x}_1$$



The deformation gradient, strain and stress tensors

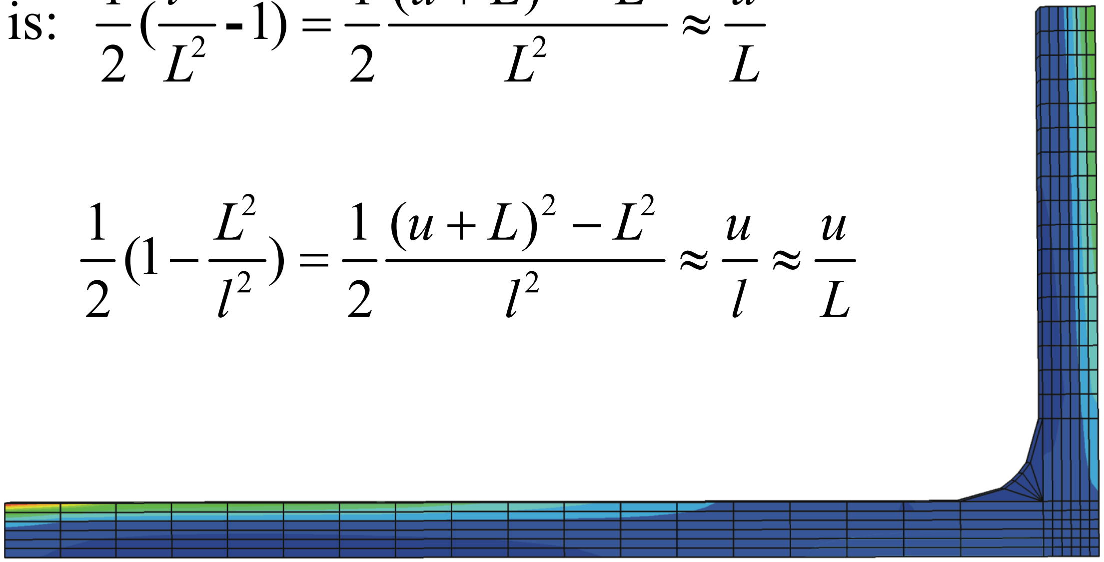
- **Lets see an example (one-dimensional)**

Green-Lagrange strains: $\mathbf{E} = \frac{1}{2}(\frac{l^2}{L^2} - 1)$

Almansi strains: $\mathbf{A} = \frac{1}{2}(1 - \frac{L^2}{l^2})$

for infinitesimal strains there is: $\frac{1}{2}(\frac{l^2}{L^2} - 1) = \frac{1}{2}\frac{(u+L)^2 - L^2}{L^2} \approx \frac{u}{L}$

and $\frac{1}{2}(1 - \frac{L^2}{l^2}) = \frac{1}{2}\frac{(u+L)^2 - L^2}{l^2} \approx \frac{u}{l} \approx \frac{u}{L}$



The deformation gradient, strain and stress tensors

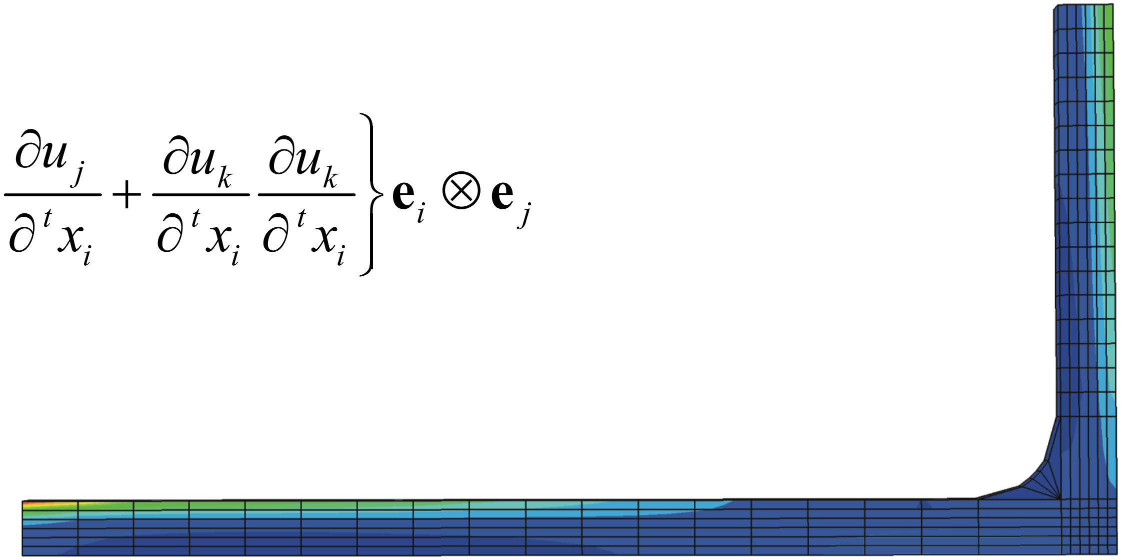
- We now consider the tensor components of the strain tensors

Green-Lagrange strains

$$\boldsymbol{\varepsilon} = \varepsilon_{ij}\mathbf{e}_i \otimes \mathbf{e}_j = \frac{1}{2}\left\{\frac{\partial u_i}{\partial\, {}^o x_j} + \frac{\partial u_j}{\partial\, {}^o x_i} + \frac{\partial u_k}{\partial\, {}^o x_i}\frac{\partial u_k}{\partial\, {}^o x_i}\right\}\mathbf{e}_i \otimes \mathbf{e}_j$$

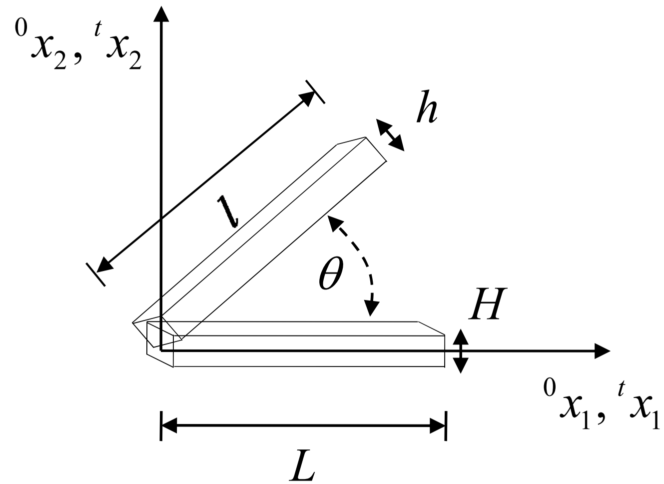
Almansi strains

$$\boldsymbol{\alpha} = \alpha_{ij}\mathbf{e}_i \otimes \mathbf{e}_j = \frac{1}{2}\left\{\frac{\partial u_i}{\partial\, {}^t x_j} + \frac{\partial u_j}{\partial\, {}^t x_i} + \frac{\partial u_k}{\partial\, {}^t x_i}\frac{\partial u_k}{\partial\, {}^t x_i}\right\}\mathbf{e}_i \otimes \mathbf{e}_j$$



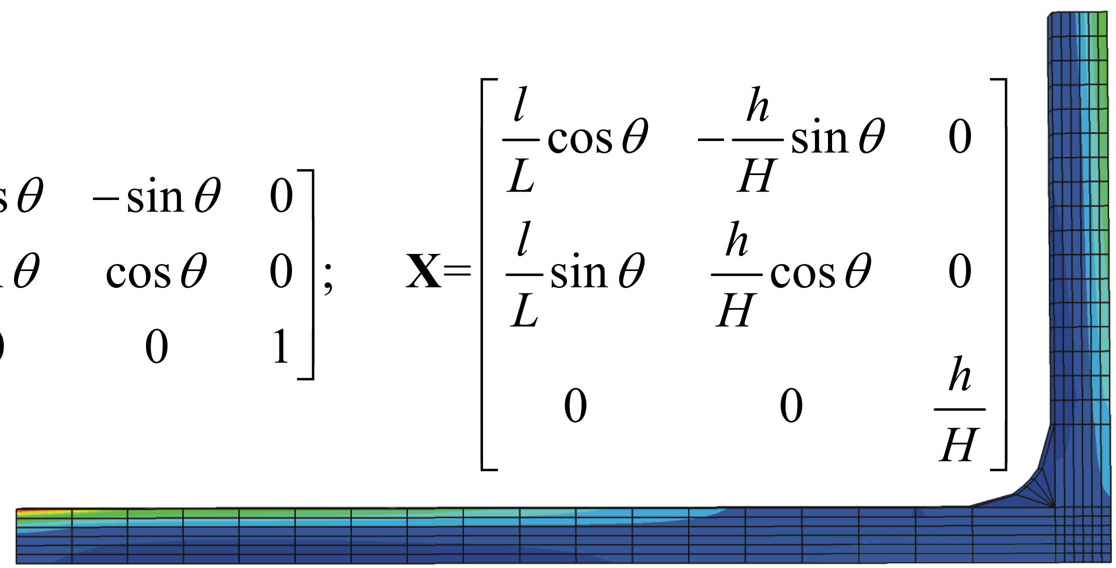
The deformation gradient, strain and stress tensors

Example – beam element



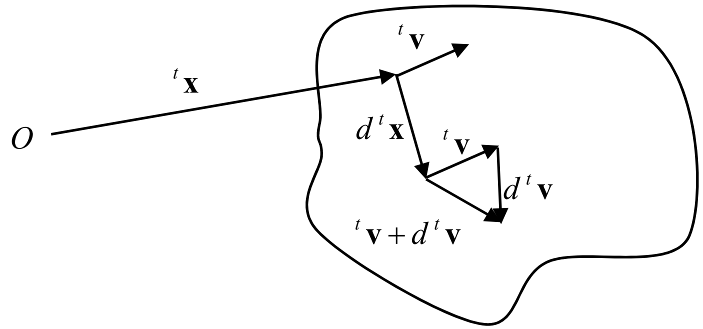
X=RU

$$\mathbf{U} = \begin{bmatrix} \frac{l}{L} & 0 & 0 \\ 0 & \frac{h}{H} & 0 \\ 0 & 0 & \frac{h}{H} \end{bmatrix}; \quad \mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad \mathbf{X} = \begin{bmatrix} \frac{l}{L}\cos\theta & -\frac{h}{H}\sin\theta & 0 \\ \frac{l}{L}\sin\theta & \frac{h}{H}\cos\theta & 0 \\ 0 & 0 & \frac{h}{H} \end{bmatrix}$$



The deformation gradient, strain and stress tensors

Now we consider the velocity gradient tensor – the difference in velocity of two points infinitesimally close



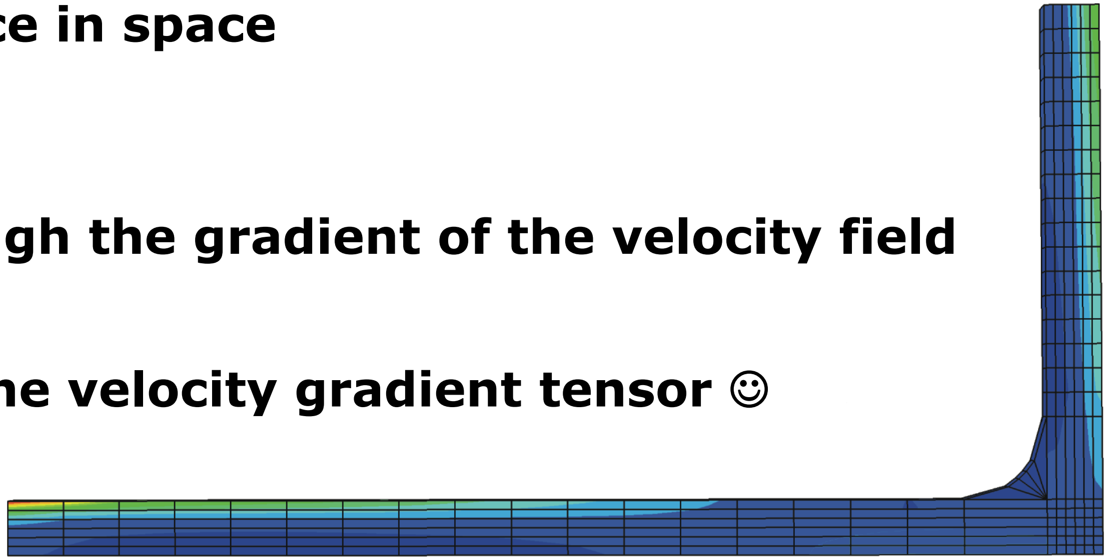
We can write change of velocity over space as a linear function of the distance in space

$$d\,{}^t\mathbf{v} = \mathbf{L}\,d\,{}^t\mathbf{x}$$

where L is given through the gradient of the velocity field at time *t*

$$\mathbf{L} = \mathbf{v} \otimes \nabla_{\mathbf{x}}$$

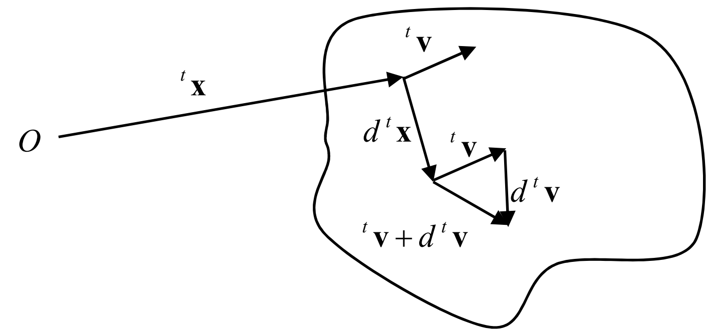
This is the velocity gradient tensor ☺



The deformation gradient, strain and stress tensors

We remember that there is:

$$d\,{}^{t}\mathbf{x} = \mathbf{X}\,d\,{}^{0}\mathbf{x}$$



which leads us to:

$$d\,{}^{t}\mathbf{v} = \dot{\mathbf{X}}\,d\,{}^{0}\mathbf{x}$$

$$\Downarrow$$

$$d\,{}^{t}\mathbf{v} = \mathbf{L}\mathbf{X}\,d\,{}^{0}\mathbf{x}$$

$$\Downarrow$$

$$\mathbf{L} = \dot{\mathbf{X}}\mathbf{X}^{-1}$$

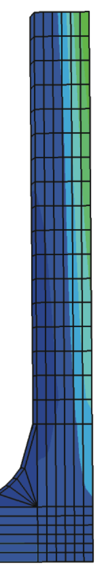
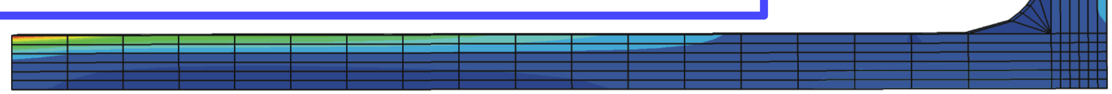
$\mathbf{L} = \mathbf{D} + \mathbf{W}$ **decomposition**

$$\mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^{T}) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\mathbf{e}_i \otimes \mathbf{e}_j$$

deformation rate tensor

$$\mathbf{W} = \frac{1}{2}(\mathbf{L} - \mathbf{L}^{T}) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right)\mathbf{e}_i \otimes \mathbf{e}_j$$

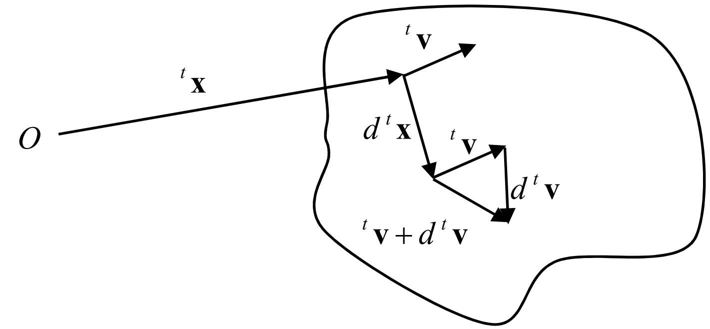
spin/rotation rate tensor

The deformation gradient, strain and stress tensors

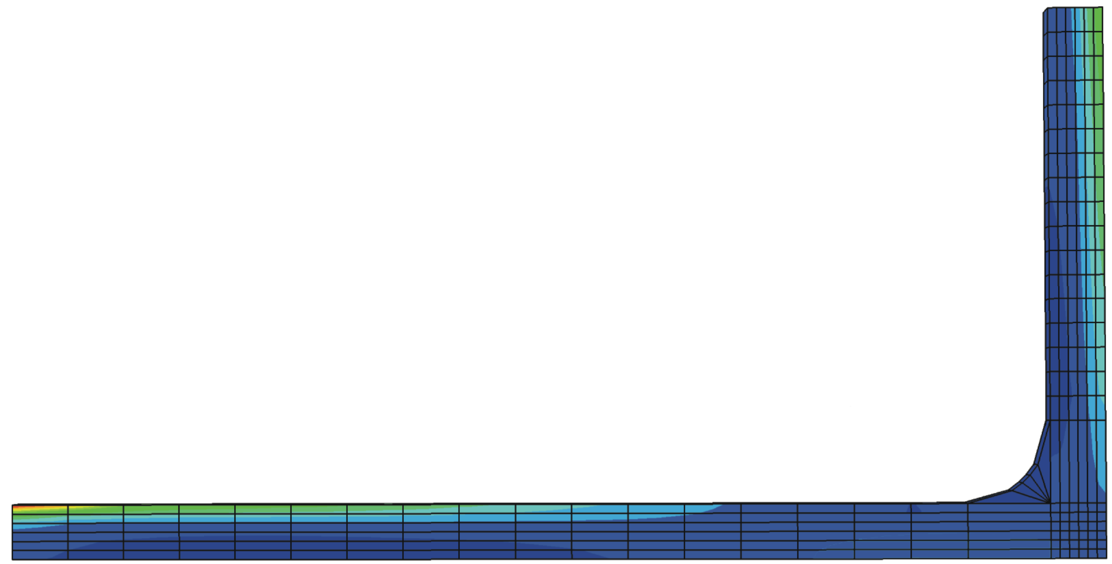
And then we may derive the Green-Lagrange velocity strain tensor

$$\dot{\boldsymbol{\varepsilon}} = {}_0^t\mathbf{X}^T \mathbf{D} \, {}_0^t\mathbf{X} \qquad \mathbf{D} = {}_t^0\mathbf{X}^T \dot{\boldsymbol{\varepsilon}} \, {}_t^0\mathbf{X}$$



We could also just have differentiated the Green-Lagrange strain tensor with resect to time

$$\dot{\boldsymbol{\varepsilon}} = \frac{1}{2}\left({}_0^t\dot{\mathbf{X}}^T \, {}_0^t\mathbf{X} + {}_0^t\mathbf{X}^T \, {}_0^t\dot{\mathbf{X}} \right)$$



The deformation gradient, strain and stress tensors

Finally we need to establish the stresses

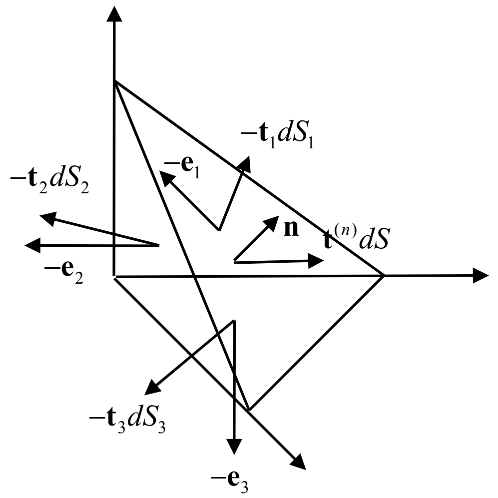
We start by introducing the Cauchy stresses:

$$\mathbf{t}^{(n)} = \boldsymbol{\tau}\mathbf{n}$$

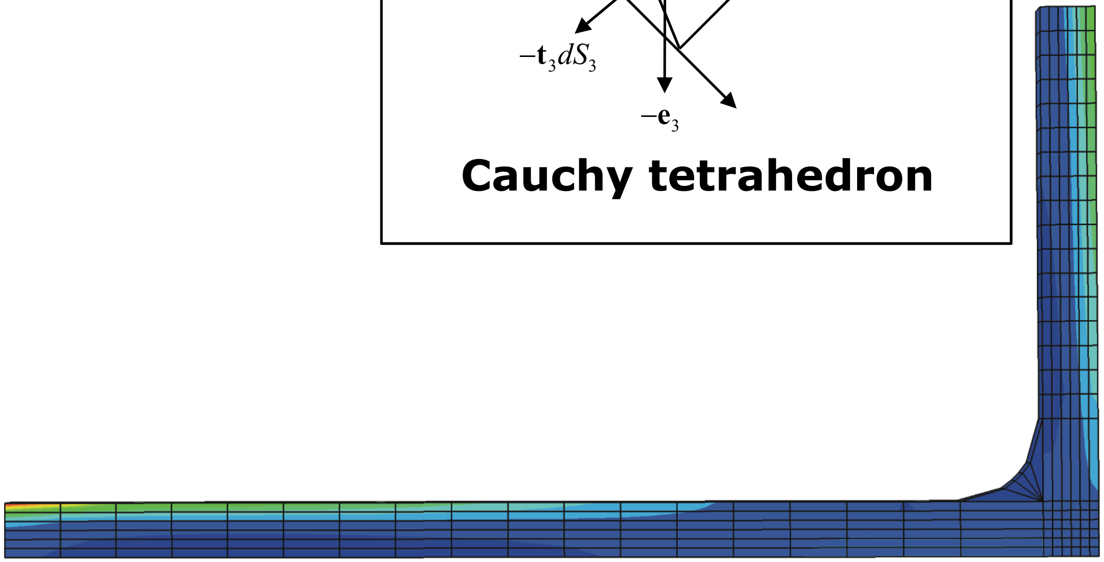
$$\mathbf{t}^{(n)} = \boldsymbol{\tau}_1\mathbf{n}_1 + \boldsymbol{\tau}_2\mathbf{n}_2 + \boldsymbol{\tau}_3\mathbf{n}_3$$

$$\begin{bmatrix} t_1^{(n)} \\ t_2^{(n)} \\ t_3^{(n)} \end{bmatrix} = \begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

Cauchy stress tensor



Cauchy tetrahedron



The deformation gradient, strain and stress tensors

Finally we introduce the second Piola-Kirchoff stresses:

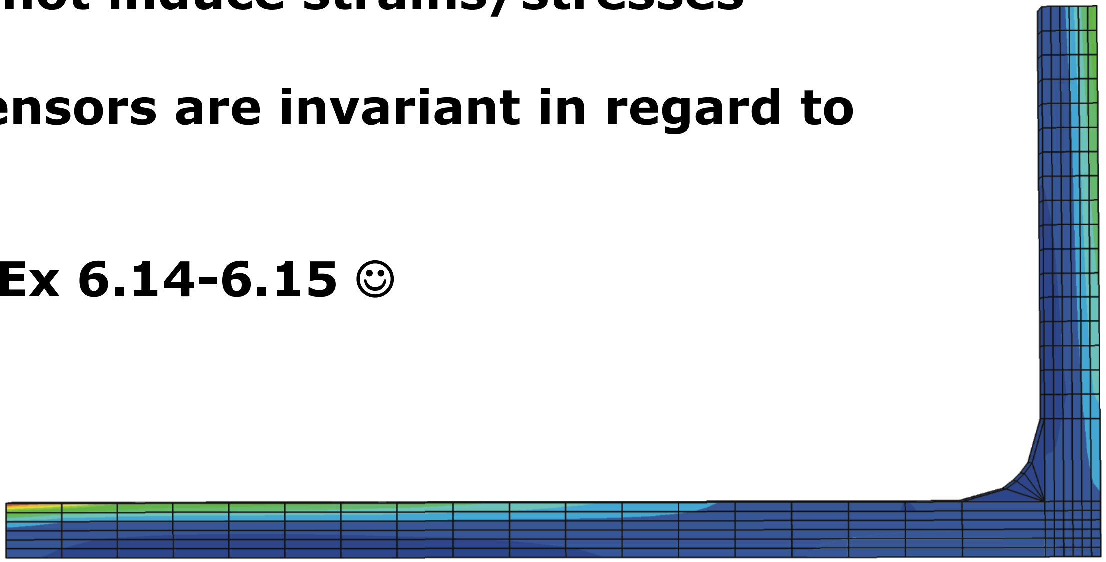
$$
{}_0^t\mathbf{S} = \frac{{}^0\rho}{{}^t\rho}\, {}_t^0\mathbf{X}\, {}^t\boldsymbol{\tau}\, {}_t^0\mathbf{X}^T
$$

these are so-called work conjugate to the Green-Lagrange strains

Rigid body motions do not induce strains/stresses

the strain and stress tensors are invariant in regard to rotations

Worthwhile to consult Ex 6.14-6.15 ☺



The deformation gradient, strain and stress tensors

We remember that we set out to solve the following equation:

$$\int_{{}^{t+\Delta t}V} {}^{t+\Delta t}\tau \delta_{t+\Delta t} e_{ij} d\, {}^{t+\Delta t}V = {}^{t+\Delta t}R$$

${}^{t+\Delta t}\tau$: Cartesian components of the Cauchy stress tensor

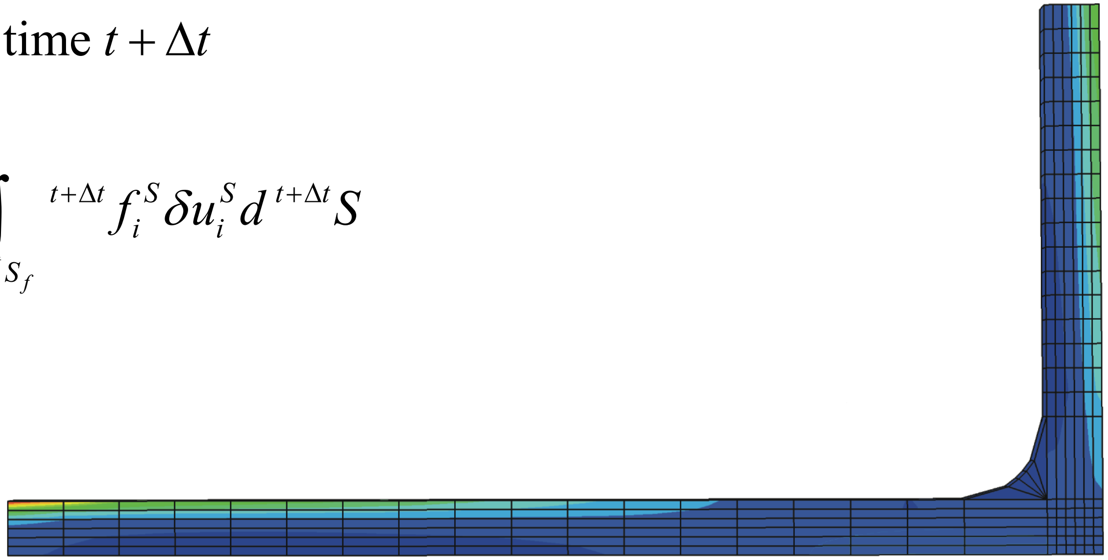
$$\delta_{t+\Delta t} e_{ij} = \frac{1}{2}\left(\frac{\partial \delta u_i}{\partial\, {}^{t+\Delta t}x_j} + \frac{\partial \delta u_j}{\partial\, {}^{t+\Delta t}x_i}\right) = \text{strain tensor corresponding to virtual displacements}$$

δu_i : Components of virtual displacement vector imposed at time $t+\Delta t$

${}^{t+\Delta t}x_i$: Cartesian coordinate at time $t+\Delta t$

${}^{t+\Delta t}V$: Volume at time $t+\Delta t$

$$^{t+\Delta t}R = \int_{{}^{t+\Delta t}V} {}^{t+\Delta t}f_i^B \delta u_i d\, {}^{t+\Delta t}V + \int_{{}^{t+\Delta t}S_f} {}^{t+\Delta t}f_i^S \delta u_i^S d\, {}^{t+\Delta t}S$$



The deformation gradient, strain and stress tensors

We remember that we set out to solve the following equation:

$$\int_{^{t+\Delta t}V} {}^{t+\Delta t}\tau \delta_{t+\Delta t} e_{ij} \, d\,{}^{t+\Delta t}V = {}^{t+\Delta t}R$$

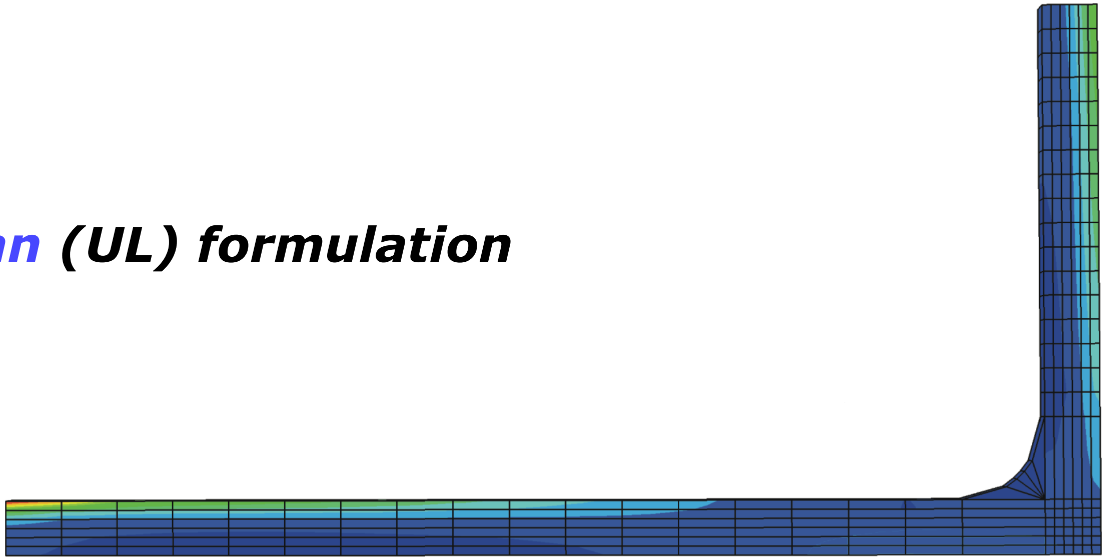
Two schemes have been formulated for this namely:

The Total Lagrangian (TL) formulation

$$\int_{^{0}V} {}^{t+\Delta t}_{\;\;0}S_{ij} \, \delta\, {}^{t+\Delta t}_{\;\;0}\varepsilon_{ij} \, d\,{}^{0}V = {}^{t+\Delta t}R$$

The Updated Lagrangian (UL) formulation

$$\int_{^{t}V} {}^{t+\Delta t}_{\;\;t}S_{ij} \, \delta\, {}^{t+\Delta t}_{\;\;t}\varepsilon_{ij} \, d\,{}^{t}V = {}^{t+\Delta t}R$$



The deformation gradient, strain and stress tensors

The resulting equations of motion for time *t* may be derived to:

The Total Lagrangian (TL) formulation

$$\int_{{}^0V} {}_0C_{ijrs}\, {}_0e_{rs}\, \delta\, {}_0e_{ij}\, d\,{}^0V + \int_{{}^0V} {}_0^tS_{ij}\, \delta\, {}_0\eta_{ij}\, d\,{}^0V = {}^{t+\Delta t}R - \int_{{}^0V} {}_0^tS_{ij}\, \delta\, {}_0e_{ij}\, d\,{}^0V$$

The Updated Lagrangian (UL) formulation

$$\int_{{}^tV} {}_0C_{ijrs}\, {}_te_{rs}\, \delta\, {}_te_{ij}\, d\,{}^tV + \int_{{}^0V} {}^t\tau_{ij}\, \delta\, {}_t\eta_{ij}\, d\,{}^tV = {}^{t+\Delta t}R - \int_{{}^tV} {}^t\tau_{ij}\, \delta\, {}_te_{ij}\, d\,{}^tV$$

Finally – in practice it is often sufficient to account for only material non-linearity

In this case the TL and the UL formulations become identical.