Method of Finite Elements II

Example 11.4, 11.5 & 11.8
and necessary theory

Fabio Wider
Centre for Mechanics

ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

IMES

$$\mathbf{K\Phi} = \lambda \mathbf{M\Phi}$$

Four groups of solution methods:

- Vector Iteration

$$\mathbf{K\Phi}_i = \lambda_i \mathbf{M\Phi}_i$$

- Transformation methods
- Polynomial iteration techniques
- Using the Sturm sequence property of characteristic polynomials

We usually use as a start vector $$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

We assume that $\mathbf{K}$ is pd and $\mathbf{y}_1 = \mathbf{M}\mathbf{x}_1$.

And by evaluating for k = 1, 2,…

$$\mathbf{K}\overline{\mathbf{x}}_{k+1} = \mathbf{y}_k$$

$$\overline{\mathbf{y}}_{k+1} = \mathbf{M}\overline{\mathbf{x}}_{k+1}$$

$$\rho(\overline{x}_{k+1}) = \frac{\overline{\mathbf{x}}_{k+1}^T \mathbf{y}_k}{\overline{\mathbf{x}}_{k+1}^T \overline{\mathbf{y}}_{k+1}}$$

$$\mathbf{y}_{k+1} = \frac{\overline{\mathbf{y}}_{k+1}}{\left(\overline{\mathbf{x}}_{k+1}^T \overline{\mathbf{y}}_{k+1}\right)^{1/2}}$$

Provided that $\mathbf{y}_1^T \mathbf{\Phi}_1 \neq 0$

$$\mathbf{y}_{k+1} \rightarrow \mathbf{M}\mathbf{\Phi}_1 \text{ and } \rho(\overline{\mathbf{x}}_{k+1}) \rightarrow \lambda_1 \text{ as } k \rightarrow \infty$$

If *l* is the last iteration, we end up having

$$\lambda_1 = \rho(\overline{\mathbf{x}}_{l+1})$$

$$\mathbf{\Phi}_1 = \frac{\overline{\mathbf{x}}_{l+1}}{\left(\overline{\mathbf{x}}_{l+1}^T \overline{\mathbf{y}}_{l+1}\right)^{1/2}}$$

… the first eigenpair!

Only thing that changes: we assume $\mathbf{M}$ to be pd and

$$\mathbf{y}_1 = \mathbf{K}\mathbf{x}_1$$

$$\mathbf{M}\overline{\mathbf{x}}_{k+1} = \mathbf{y}_k$$

$$\overline{\mathbf{y}}_{k+1} = \mathbf{K}\overline{\mathbf{x}}_{k+1}$$

$$\rho(\overline{x}_{k+1}) = \frac{\overline{\mathbf{x}}_{k+1}^T\,\overline{\mathbf{y}}_k}{\overline{\mathbf{x}}_{k+1}^T\,\mathbf{y}_{k+1}}$$

$$\mathbf{y}_{k+1} = \frac{\overline{\mathbf{y}}_{k+1}}{\left(\overline{\mathbf{x}}_{k+1}^T\,\mathbf{y}_{k+1}\right)^{1/2}}$$

Provided that $\mathbf{\Phi}_n^T \mathbf{y}_1 \neq 0$

$$\mathbf{y}_{k+1} \rightarrow \mathbf{K}\mathbf{\Phi}_n \text{ and } \rho(\overline{\mathbf{x}}_{k+1}) \rightarrow \lambda_n \text{ as } k \rightarrow \infty$$

and if *l* is again the last iteration, we have

$$\lambda_n = \rho(\overline{\mathbf{x}}_{l+1})$$

$$\mathbf{\Phi}_n = \frac{\overline{\mathbf{x}}_{l+1}}{\left(\overline{\mathbf{x}}_{l+1}^T \mathbf{y}_l\right)^{1/2}}$$

…the last eigenpair!

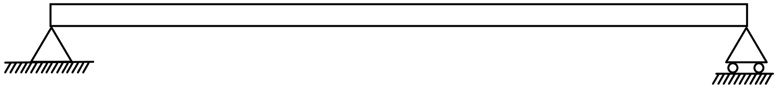
In both iteration procedures the convergence is measured by

$$\frac{\left|\lambda_l^{(k+1)} - \lambda_l^{(k)}\right|}{\lambda_l^{(k+1)}} \leq tol$$

Where tol is usually some decimals, e.g. 1e-8

Use forward iteration with *tol* = 1e-6 to evaluate λ_4 and $\mathbf{\Phi}_4$ of the eigenproblem $\mathbf{K\Phi} = \lambda \mathbf{M\Phi}$, where

$$\mathbf{K} = \begin{pmatrix} 5 & -4 & 1 & 0 \\ -4 & 6 & -4 & 1 \\ 1 & -4 & 6 & -4 \\ 0 & 1 & -4 & 5 \end{pmatrix};\ \mathbf{M} = \begin{pmatrix} 2 & & & \\ & 2 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$



Using the starting vector $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Using MATLAB, we get the following fourth eigenpair…

k	$\bar{\mathbf{x}}_{k+1}$	$\bar{\mathbf{y}}_{k+1}$	$\rho(\bar{\mathbf{x}}_{k+1})$	$\mathbf{y}_{k+1}$	$\dfrac{\lvert\lambda_4^{(k+1)} - \lambda_4^{(k)}\rvert}{\lambda_4^{(k+1)}}$
10	−1.1416	−24.3237	10.63845	−2.2864	0.0000009437
	2.7170	57.8405		5.4369	
	−7.7476	−82.4219		−7.7476	
	5.9816	63.6157		5.9798	

$$\lambda_4 \doteq 10.63845; \qquad \boldsymbol{\phi}_4 \doteq \begin{bmatrix} -0.10731 \\ 0.25539 \\ -0.72827 \\ 0.56227 \end{bmatrix}$$

Improving convergence rate?

What when $\mathbf{M}$ and $\mathbf{K}$ are not pd?

Applying a shift μ:

$$\left(\mathbf{K}-\mu\mathbf{M}\right)\mathbf{\Phi}=\eta\mathbf{M}\mathbf{\Phi}$$

Use inverse iteration in order to calculate (λ_1, $\mathbf{\Phi}_1$) of the problem $\mathbf{K\Phi} = \lambda \mathbf{M\Phi}$, where $\mathbf{K}$ and $\mathbf{M}$ are given in Example 11.4. Then impose a shift μ = 10 and show that in the inverse iteration convergence occurs toward (λ_4, $\mathbf{\Phi}_4$). Use again a tolerance of 1e-6.

After three iterations we get for the first eigenpair…

$$\lambda_1 \doteq 0.09654; \qquad \boldsymbol{\phi}_1 \doteq \begin{bmatrix} 0.3126 \\ 0.4955 \\ 0.4791 \\ 0.2898 \end{bmatrix}$$

Then imposing a shift of μ = 10, we obtain…

$$\mathbf{K} - \mu\mathbf{M} = \begin{pmatrix} -15 & -4 & 1 & 0 \\ -4 & -14 & -4 & 1 \\ 1 & -4 & -4 & -4 \\ 0 & 1 & -4 & -5 \end{pmatrix}$$

Using now again the inverse iteration on the problem $(\mathbf{K}-\mu\mathbf{M})\mathbf{\Phi}=\eta\mathbf{M}\mathbf{\Phi}$

we obtain convergence after six iterations and get…

$$\rho(\overline{\mathbf{x}}_7) = 0.6385; \qquad \mathbf{x}_7 = \begin{bmatrix} -0.1076 \\ 0.2556 \\ -0.7283 \\ 0.5620 \end{bmatrix}$$

Since we imposed a shift, we know:

- $\mu + \rho(\bar{\mathbf{x}}_7)$ is approximation to an eigenvalue
- $\mathbf{x}_7$ is approximation to the corresponding eigenvector

We do not know:

- which eigenpair is approximated

Solution here: by comparing with Example 11.4, we see that we have the fourth eigenpair.

In case that no other example is available: Do not choose shift arbitrarliy, use e.g. Rayleigh Quotient Iteration. (Bathe 11.2.4)

Orthogonalize starting vector to eigenvectors already calculated

→ Convergence occurs only to other eigenvectors

Suppose:

- Calculated are eigenvectors $\boldsymbol{\Phi}_1, \boldsymbol{\Phi}_2, ..., \boldsymbol{\Phi}_m$
- Now $\mathbf{M}$-orthogonalize $\mathbf{x}_1$ to these eigenvectors (deflate $\mathbf{x}_1$)

$$\tilde{\mathbf{x}}_1 = \mathbf{x}_1 - \sum_{i=1}^{m} \alpha_i \boldsymbol{\Phi}_i \qquad \text{where} \qquad \alpha_i = \boldsymbol{\Phi}_i^T \mathbf{M} \mathbf{x}_1;\ i = 1, ..., m$$

In inverse iteration $\tilde{\mathbf{x}}_1$ is now used as a starting vector instead of $\mathbf{x}_1$.

Convergence occurs now to eigenpair (λ_{m+1}, $\mathbf{\Phi}_{m+1}$), which is unknown yet.

Calculate, using Gram-Schmidt orthogonalization, an appropriate starting iteration vector for the solution of the problem $\mathbf{K\Phi} = \lambda \mathbf{M\Phi}$, where $\mathbf{K}$ and $\mathbf{M}$ are given in Example 11.4. Assume that the eigenpairs (λ_1, $\mathbf{\Phi}_1$) and (λ_4, $\mathbf{\Phi}_4$), are known as obtained in Example 11.5 and that convergence to another eigenpair is sought.

Deflate unit full vector of the known eigenvectors:

$$\tilde{\mathbf{x}}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \alpha_1 \mathbf{\Phi}_1 - \alpha_4 \mathbf{\Phi}_4$$

where the *α*'s are obtained by $\alpha_1 = \mathbf{\Phi}_1^T \mathbf{M} \mathbf{x}_1;\ \alpha_4 = \mathbf{\Phi}_4^T \mathbf{M} \mathbf{x}_1$

Substituting the values obtained in Example 11.5 leads us to…

$$\alpha_1 = 2.385;\ \alpha_2 = 0.1299$$

So we finally end up having…

$$\tilde{\mathbf{x}}_1 = \begin{pmatrix} 0.2683 \\ -0.2149 \\ -0.04812 \\ 0.2358 \end{pmatrix}$$

Thank you for your attention.