Solution of dynamic non-linear problems

Example 9.12.

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Time integration methods

1. **Explicit methods:**
   - Solution of equilibrium equations at time $t + \Delta t$
   - No factorization of stiffness matrix
   - Conditional stability
   - E.g. central difference method

2. **Implicit methods:**
   - Solution for equilibrium equations at time t
   - Factorization of stiffness matrix
   - Unconditional stability
   - E.g. Newmark method, Wilson θ method

Stability

Stability of an integration method means that numerical errors present in the solution for any initial conditions do not amplify during the integration.

Let be $\rho(\mathbf{A})$ be the spectral radius defined as

$$\rho(\mathbf{A}) = \max_{i=1,2,\cdots} |\lambda_i| \,. \tag{1}$$

Stability criterion (SC):

1) If all eigenvalues of $\mathbf{A}$ are distinct, we must have $\rho(\mathbf{A}) \leqslant 1$.
2) If $\mathbf{A}$ contains multiple eigenvalues, we require that all eigenvalues be (in modulus) smaller than 1.

A time integration method is

- **conditional stable** if SC holds for any initial conditions and any time step $\Delta t \leq \Delta t_{crit}$.
- **unconditional stable** if SC holds for any initial conditions and any time step Δt.

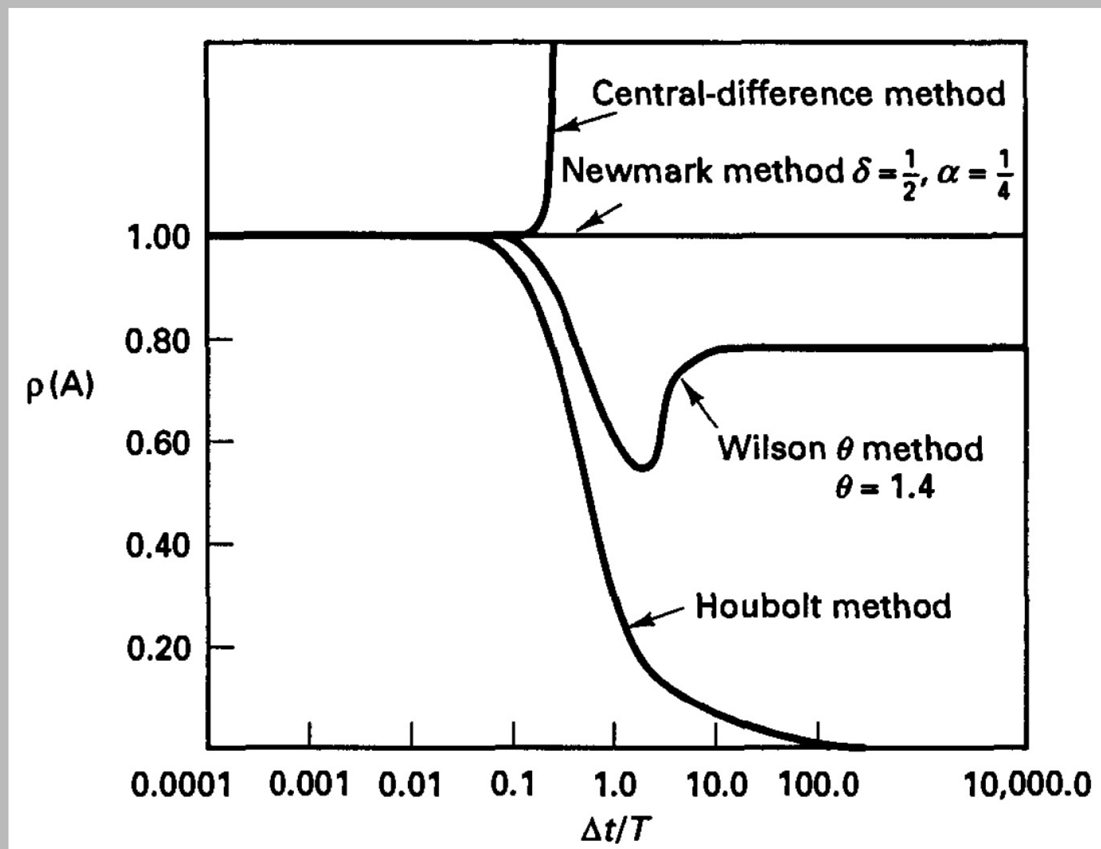

Fig. 1 Spectral radius of approximation operators, case $\xi = 0.0$ (from Bathe [1996]).

Accuracy

Initial value problem:

$$\ddot{x} + \omega x = 0 \quad \text{with} \quad {}^{0}x = 1.0; \quad {}^{0}\dot{x} = 0.0; \quad {}^{0}\ddot{x} = -\omega^2. \tag{2}$$

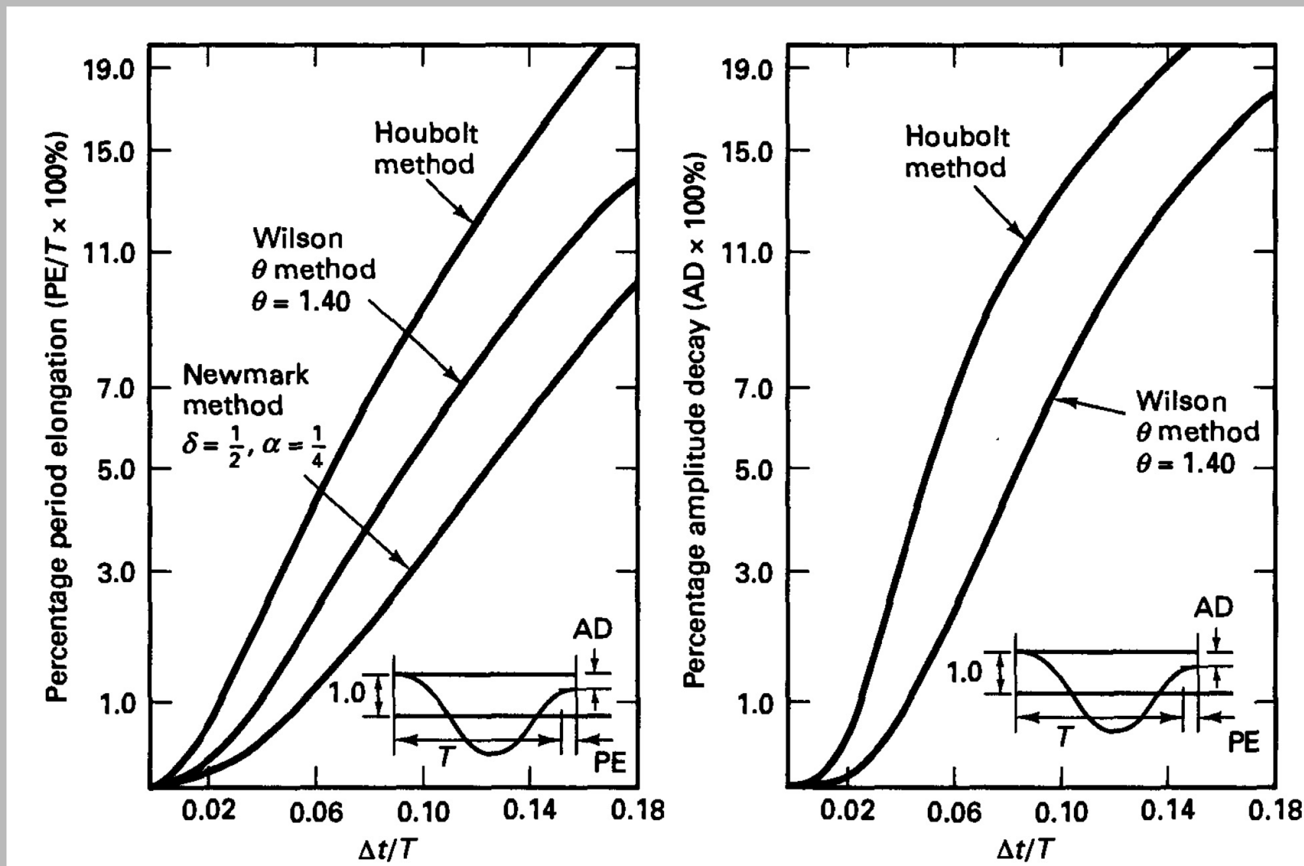

Fig. 2 Percentage period elongations and amplitude decays (from Bathe [1996]).

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Analyze the central difference method for its integration stability. Assume that damping effects can be neglected.

Recursive relationship for solution vector:

$$\begin{bmatrix} {}^{t+\Delta t}x \\ {}^{t}x \end{bmatrix} = \mathbf{A} \begin{bmatrix} {}^{t}x \\ {}^{t-\Delta t}x \end{bmatrix}, \tag{3}$$

$$\text{where } \mathbf{A} = \begin{bmatrix} 2-\omega^2\Delta t^2 & -1 \\ 1 & 0 \end{bmatrix}. \tag{4}$$

Eigenvalue problem: $\mathbf{A}\mathbf{u} = \lambda\mathbf{u}$

Characteristic equation:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \left(2 - \omega^2\Delta t^2 - \lambda\right)(-\lambda) + 1 = 0\,. \tag{5}$$

Eigenvalues:

$$\lambda_1 = \frac{2 - \omega^2\Delta t^2}{2} + \sqrt{\frac{\left(2 - \omega^2\Delta t^2\right)^2}{4} - 1}\,, \tag{6}$$

$$\lambda_2 = \frac{2 - \omega^2\Delta t^2}{2} - \sqrt{\frac{\left(2 - \omega^2\Delta t^2\right)^2}{4} - 1}\,. \tag{7}$$

SC is satisfied for $\Delta t/T \leqslant 1/\pi$. Hence, the central difference method is stable provided that $\Delta t \leq \Delta t_{crit}$, where $\Delta t_{crit} \leqslant T/\pi$.